PROBABILITY DISTRIBUTION:

BINOMIAL DISTRIBUTION : (DISCRETE)

Each trial has two possible outcomes, generally called success and failure. Such a trial is known as Bernoulli trial. An experiment consisting of a repeated number of Bernoulli trials is called Binomial experiment.

A Binomial experiment must possess the following properties :

(i) There must be a fixed number of trials.

(ii) All trials must have identical probabilities of success.

(iii) The trials must be independent of each other.

Defn:

A random variable x is said to follow Binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$p(x=x) = nc_x \, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

Here n – number of trials.

p – probability of success.

q – probability of failure

x – A random variable which represents the number of success.

Find the moment generating function, its mean and variance using Binomial distribution.

Moment generating function:

The MGF of a Binomial variate is

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} nc_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nc_x (e^t p)^x q^{n-x}$$

$$= nc_0 (e^t p)^0 q^n + nc_1 (e^t p)^1 q^{n-1} + nc_2 (e^t p)^2 q^{n-2} + \cdots + nc_n (e^t p)^n q^{n-n}$$

$$= q^n + nc_1 (e^t p) q^{n-1} + nc_2 (e^t p)^2 q^{n-2} + \cdots + (e^t p)^n$$

$$M_x(t) = (q + e^t p)^n$$

Mean:

$$E(x) = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}(q+pe^t)^n\right]_{t=0}$$

$$= \left[n(q+pe^t)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= n(q+pe^0)^{n-1} p$$

$$= np(q+p)^{n-1} \quad [\because q+p=1]$$

$$E(x) = np$$

Now $E(x^2) = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$

$$= \left[\frac{d^2}{dt^2}(q+pe^t)^n\right]_{t=0}$$

$$= \left\{\frac{d}{dt}\left[n(q+pe^t)^{n-1} pe^t\right]\right\}_{t=0}$$

$$= \left[np\left[e^t(n-1)(q+pe^t)^{n-2} pe^t + (q+pe^t)^{n-1} e^t\right]\right]_{t=0}$$

$$= \left[np\left(e^0(n-1)(q+pe^0)^{n-2} pe^0 + (q+pe^0)^{n-1} e^0\right)\right]$$

$$= [np[(n-1)p+1]]$$

$$= n(n-1)p^2 + np$$

$$= n^2p^2 - np^2 + np$$

Variance:

$$Var(x) = E(x^2) - (E(x))^2$$

$$= n^2p^2 - np^2 + np - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= -np^2 + np$$

$$= np(1-p) = npq$$

$$\therefore Var(x) = npq$$

(1) Find the Binomial distribution for which the mean is 4 and variance is 3.

Soln: Given: Mean : $np = 4$

Variance : $npq = 3$

Consider $npq = 3$

$(4)q = 3$

$q = 3/4$

WKT $p+q = 1$

$\Rightarrow p = 1-q$

$\Rightarrow p = 1 - 3/4 = 1/4$

$\Rightarrow p = 1/4$

Now, $np = 4$ (given)

$\Rightarrow n(1/4) = 4$

$\Rightarrow n = 16$

$\therefore p(x=x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \ldots n$

$= 16C_x (1/4)^x q^{16-x}, \quad x = 0, 1, 2 \ldots 16$

(2) Find the Binomial distribution for which the mear is 6 and 2 respectively. Find $p(x \geq 1)$

Soln: Given: Mean: $np = 6$

Variance: $npq = 2$

Consider $npq = 2$

$(6)q = 2 \Rightarrow q = 1/3$

WKT $p+q = 1 \Rightarrow p = 1-q$

$= 1 - 1/3 = 2/3$

$\Rightarrow p = 2/3$

Now, $np = 6$ (given)

$n(2/3) = 6$

$\Rightarrow n = 9$

$\therefore p(x = x) = 9C_x (2/3)^x (1/3)^{9-x}, \quad x = 0, 1, 2 \dots, 9$

Now $p(x \geq 1) = 1 - p(x < 1)$

$= 1 - p(x = 0)$

$= 1 - 9C_0 (2/3)^0 (1/3)^{9-0}$

$= 1 - (1/3)^9$

$= 0.9999$

Find the Binomial distribution, if x is a Binomial variate with mean $= 4$ and standard deviation is $\sqrt{2}$

Soln:

Mean: $np = 4$

S.D: $\sqrt{\text{variance}}$: $\sqrt{npq} = \sqrt{2}$

$\Rightarrow npq = 2$

Consider $npq = 2$

$(4)\, q = 2$

$q = 1/2$

WKT $p+q = 1 \Rightarrow p = 1-q$

$\Rightarrow p = 1/2$

Consider $np = 4$

$\Rightarrow n(1/2) = 4$

$\Rightarrow n = 8$

$\therefore P(X=x) = nC_x\, p^x q^{n-x}$, $x = 0,1,2,\ldots,n$

$= 8C_x\, (1/2)^x (1/2)^{8-x}$, $x = 0,1,2,\ldots,8$

(4) Check whether the following data follow Binomial distribution or not. Mean = 3 ; Variance = 4.

Soln: Mean : $np = 3$

Variance : $npq = 4$

Consider $npq = 4$

$(3)q = 4$

$q = 4/3$

Since numerator is greater than denominator and $p+q \neq 1$, it is not a Binomial distribution.

(5) If x is a Binomial variate with $n=6$ and $9P(x=4)=P(x=2)$ Find Binomial distribution.

Soln: Given: $n=6$

WKT $P(X=x) = nC_x p^x q^{n-x}, \; x=0,1,2,\ldots,n$

Given: $9P[X=4] = P[X=2]$

Given: $9P[X=4] = P[X=2]$

$$9[6C_4 p^4 q^{6-4}] = 6C_2 p^2 q^{6-2}$$

$$9[\cancel{15} p^{\cancel{4}\,2} \cancel{q^2}] = \cancel{15} \cancel{p^2} q^{\cancel{4}\,2}$$

$$9p^2 = q^2$$

$$9p^2 - (1-p)^2 = 0$$

$$9p^2 - [1-2p+p^2] = 0$$

$$9p^2 - 1 + 2p - p^2 = 0$$

$$8p^2 + 2p - 1 = 0$$

$$8p^2 + 4p - 2p - 1 = 0$$

$$4p[2p+1] - [2p+1] = 0$$

$$\Rightarrow [2p+1][4p-1] = 0$$

$$\Rightarrow p = -1/2 \text{ (or) } 1/4$$

$p = -1/2$ is impossible, since p should be btwn 0 & 1

$$\Rightarrow p = 1/4$$

WKT $p+q=1$

$$q = 1-p = 1-1/4 = 3/4$$

$$q = 3/4$$

$$\therefore \; p(X=x) = nC_x \, p^x q^{n-x}, \; x=0,1,\ldots n$$

$$= 6C_x (1/4)^x (3/4)^{6-x}, \; x=0,1,\ldots,6$$

6) In a large consignment of electric bulbs 10% are defective. A random sample of 20 is taken for inspection. Find the probability that

(i) Atmost there are 3 defective bulbs

(ii) Exactly there are 3 defective bulbs

(iii) All are good bulbs.

(iv) Atleast there are 3 defective bulbs.

Soln:

Here $n=20$; $p = 10\% = \frac{10}{100} = 0.1$

$$q = 1-p = 1-0.1 = 0.9$$

$$p(X=x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

$$= 20C_x (0.1)^x (0.9)^{20-x}, \quad x = 0, 1, 2, \ldots 20$$

(i) p(Atmost there are three (3) defective bulbs)

$$p(X \le 3) = p(X=0) + p(X=1) + p(X=2) + p(X=3)$$

$$= 20C_0 (0.1)^0 (0.9)^{20} + 20C_1 (0.1)^1 (0.9)^{19} + 20C_2 (0.1)^2 (0.9)^{18} + 20C_3 (0.1)^3 (0.9)^{17}$$

$$= 0.8670$$

(ii) p(Exactly 3 defective bulbs)

$$p(X=3) = 20C_3 (0.1)^3 (0.9)^{17}$$

$$= 0.1901.$$

(iii) p(All are good bulbs)

$$p(\text{None are defective}) = p(X=0)$$

$$= 20C_0 (0.1)^0 (0.9)^{20}$$

$$= 0.1216$$

(iv) p(Atleast there are 3 defective bulbs)

$$p(X \ge 3) = 1 - p(X < 3) = 1 - [p(X=0) + p(X=1) + p(X=2)]$$

$$= 1 - [20C_0 (0.1)^0 (0.9)^{20} + 20C_1 (0.1)^1 (0.9)^{19} + 20C_2 (0.1$$

$$= 0.3231$$

(7) Four coins are tossed simultaneously. what is the probability of getting (i) 2 heads (ii) atleast 2 heads (iii) atmost 2 heads.

Soln:

WKT $p(X=x) = nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, \dots, n$

Here p = prob. of success = $p(\text{Head}) = 1/2$

q = prob. of Failure = $p(\text{Tail}) = 1/2$

Given: $n = 4$

(i) $p(2 \text{ heads}) = p(x=2)$

$$= 4C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{4-2}$$

$$= 4C_2 \left(\tfrac{1}{2}\right)^4$$

$$= 3/8$$

(ii) $p(\text{Atleast } 2 \text{ heads}) = p(x \geq 2)$

$$= p(x=2) + p(x=3) + p(x=4)$$

$$= 4C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{4-2} + 4C_3 \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^{4-3} + 4C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^{4-4}$$

$$= 11/16$$

(iii) $p(\text{Atmost } 2 \text{ heads}) = p(x \leq 2)$

$$= p(x=0) + p(x=1) + p(x=2)$$

$$= 4C_0 (1/2)^0 (1/2)^{4-0} + 4C_1 (1/2)^1 (1/2)^{4-1} + 4C_2 (1/2)^2 (1/2)^{4-2}$$

$$= 11/16$$

(18) 12 coins are thrown 256 times. Find the number of trials which may result in 8 heads?

Soln: Given: $n = 12$ and $N = 256$

x : Number of heads $= 8$

$p =$ probability of success $= p(\text{Head}) = 1/2$

$q =$ probability of failure $= p(\text{Tail}) = 1/2$

$$p(x=x) = nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

$$= 12C_x (1/2)^x (1/2)^{12-x}, \quad x = 0, 1, \ldots 12$$

$$p(x=8) = 12C_8 \left(\tfrac{1}{2}\right)^8 \left(\tfrac{1}{2}\right)^{12-8}$$
$$= 12C_8 \left(\tfrac{1}{2}\right)^{12}$$
$$= 495 \left(\tfrac{1}{2}\right)^{12}$$

$\therefore$ No. of trials which results in 8 heads out of 256 trials $= 256 \times \frac{495}{2^{12}} = \frac{126,720}{2^{12}} = 30.93 \simeq 31$

19) 6 dice are thrown 729 times. How many times do you expect atleast 3 dice to show a five or six?

Soln: Given $n=6$ and $N=729$

p: probability of getting 5 or 6 with in one die

$$= P(5 \text{ or } 6) = p(5)+p(6) = \frac{1}{6}+\frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$q=$ probability of failure $= 1-p = 1-\frac{1}{3} = \frac{2}{3}$

$$P(x=x) = nC_x\, p^x q^{n-x}, \quad x=0,1,2,\ldots,n$$
$$= 6C_x \left(\tfrac{1}{3}\right)^x \left(\tfrac{2}{3}\right)^{6-x}, \quad x=0,1,\ldots 6$$

p(atleast 3 dice to show a 5 or 6)

$$p(x \geq 3) = p(3) + p(4) + p(5) + p(6)$$

(or) $p(x \geq 3) = 1 - p(x < 3)$

$$= 1 - [p(0) + p(1) + p(2)]$$

$$= 1 - \left[6c_0 (1/3)^0 (2/3)^6 + 6c_1 (1/3)^1 (2/3)^5 + 6c_2 (1/3)^2 (2/3)^4\right]$$

$$= 1 - \left[(2/3)^6 + 6(1/3)(2/3)^5 + \frac{6 \times 5}{1 \times 2} (1/3)^2 (2/3)^4\right]$$

$$= 1 - [0.6803]$$

$$= 0.3196$$

No. of trials which results in 5 or 6 in dice out of 729 times $= 0.3196 \times 729 = 232.98 \simeq 233.$

20) A MGF of random variable x is $(\frac{1}{2}+\frac{1}{3}e^{t})^{9}$. Find mean and variance. Find $p(x \le 2)$, $p(x=0)$.

Soln:

WKT $M_x(t) = (q+pe^t)^n$

Given: $M_x(t) = (\frac{1}{2}+\frac{1}{3}e^t)^9$

$\Rightarrow q = \frac{1}{2}$; $p = \frac{1}{3}$; $n = 9$

$\therefore$ $p(x=x) = 9C_x (\frac{1}{3})^x (\frac{1}{2})^{9-x}$, $x = 0, 1, \dots, 9$

$\therefore$ Mean $= np = 9 \times \frac{1}{3} = 3$

Variance $= npq = 9 \times \frac{1}{3} \times \frac{1}{2} = \frac{3}{2}$

$$\begin{aligned} p(x \le 2) &= p(x=0) + p(x=1) + p(x=2) \\ &= 9C_0 (\tfrac{1}{3})^0 (\tfrac{1}{2})^9 + 9C_1 (\tfrac{1}{3})^1 (\tfrac{1}{2})^8 + 9C_2 (\tfrac{1}{3}) (\tfrac{1}{2})^7 \\ &= (\tfrac{1}{2})^9 + 9 (\tfrac{1}{3}) (\tfrac{1}{2})^8 + \frac{9 \times 8}{1 \times 7} (\tfrac{1}{3})^2 (\tfrac{1}{2})^7 \\ &= 0.0449 \end{aligned}$$

$$\begin{aligned} p(x=0) &= 9C_0 (\tfrac{1}{3})^0 (\tfrac{1}{2})^9 \\ &= (\tfrac{1}{2})^9 \\ &= 0.0019 \end{aligned}$$