Conditional probability:

The conditional probability of an event B assuming that the event A has happened, is defined as,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

Similarly,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0$$

Theorem:

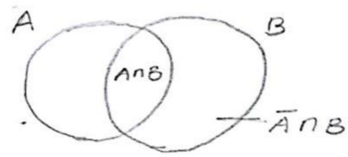
If A and B are independent, then prove that

1. $\bar{A}$ and B are independent
2. A and $\bar{B}$ are independent
3. $\bar{A}$ and $\bar{B}$ are independent.

Proof: Given A and B are independent $\Rightarrow P(A \cap B) = P(A)P(B)$ $\longrightarrow (1)$

1. $P(\bar{A} \cap B) = P(\bar{A}) \cdot P(B)$

Consider,



$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

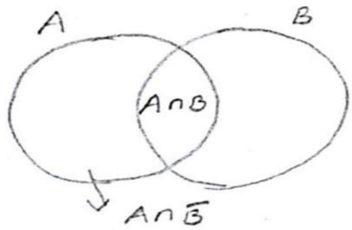
$$= P(B) - P(A)P(B) \quad (\text{from } (1))$$

$$= P(B)[1 - P(A)]$$

$$\boxed{P(\bar{A} \cap B) = P(B) \cdot P(\bar{A})}$$

$\therefore \bar{A}$ and B are independent.

2. $P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$



Consider,

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)]$$

$$= P(A \cap \bar{B}) + P(A \cap B)$$

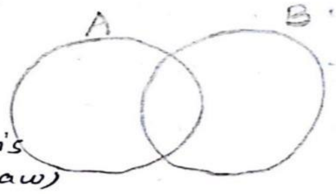
$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A)P(B)$$

$$= P(A)[1 - P(B)]$$

$$\boxed{P(A \cap \bar{B}) = P(A)\,P(\bar{B})}$$ $\therefore$ A and $\bar{B}$ are independen

3. $P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$



Consider,

$$P(\bar{A} \cap \bar{B}) = \overline{P(A \cup B)} \quad \text{(by Demorgan's law)}$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [P(A) + P(B) - P(A) \cdot P(B)]$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$

$$= P(\bar{A}) - P(B)[1 - P(A)]$$

$$= P(\bar{A}) - P(B) \cdot P(\bar{A})$$

$$= P(\bar{A})[1 - P(B)]$$

$$\boxed{P(\bar{A} \cap \bar{B}) = P(\bar{A})\,P(\bar{B})}$$

$\therefore$ $\bar{A}$ and $\bar{B}$ are independent.

$$= P(\bar{A}) - P(B)\,P(\bar{A})$$

$$= P(\bar{A})\,[1 - P(B)]$$

$$P(\bar{A} \cap \bar{B}) = P(\bar{A})\,P(\bar{B})$$

Hence $\bar{A}$ and $\bar{B}$ are independent.

Hence proved

7. A bag contains 3 Red and 4 white balls. Two balls are drawn without replacement. What is the probability that both balls are red?

Soln.:

Drawing a Red ball in the 1st draw is

$$P(A) = \frac{3C_1}{7C_1} = \frac{3}{7}$$

Drawing a Red ball in second draw given that first drawn is Red ball $= P(B/A)$

$$\therefore P(B/A) = \frac{2C_1}{6C_1} = \frac{1}{3}$$

we know that

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(B/A)\,P(A)$$

$$= \frac{1}{3} \times \frac{3}{7}$$

$$P(A \cap B) = \frac{1}{7}$$