



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



Continuous Two Dimensional Random Variables.

1. Given  $f(x, y) = \begin{cases} cx(x-y) & , 0 < x < a, -x < y < x \\ 0 & \text{otherwise} \end{cases}$

Find i).  $c$  ii).  $F(x)$  iii).  $F(y/x)$

Soln.

$$i). \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$$

$$\int_0^a \int_{-x}^x cx(x-y) dy dx = 1$$

$$c \int_0^a \int_{-x}^x (x^2 - xy) dy dx = 1$$

$$c \int_0^a \left[ x^2 y - x \frac{y^2}{2} \right]_{y=-x}^x dx = 1$$

$$c \int_0^a \left[ \left( x^3 - \frac{x^3}{2} \right) - \left( -x^3 - \frac{x^3}{2} \right) \right] dx = 1$$

$$c \int_0^a \left[ x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2} \right] dx = 1$$

$$c \int_0^a 2x^3 dx = 1$$

$$2c \left( \frac{x^4}{4} \right)_0^a = 1$$

$$\frac{c}{2} (a^4 - 0) = 1$$

$$\frac{16c}{2} = 1$$

$$8c = 1 \Rightarrow c = \frac{1}{8}$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



$$\therefore F(x, y) = \begin{cases} \frac{1}{8} x(x-y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise.} \end{cases}$$

ii). Marginal density function of  $x$  (MDF of  $x$ )

$$F(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$= \int_{-x}^x \frac{1}{8} x(x-y) dy$$

$$= \frac{1}{8} \int_{-x}^x (x^2 - xy) dy$$

$$= \frac{1}{8} \left[ x^2 y - x \frac{y^2}{2} \right]_{y=-x}^x$$

$$= \frac{1}{8} \left[ \left( x^3 - \frac{x^3}{2} \right) - \left( -x^3 - \frac{x^3}{2} \right) \right]$$

$$= \frac{1}{8} \left[ x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2} \right]$$

$$= \frac{2x^3}{8}$$

$$F(x) = \frac{x^3}{4}, \quad 0 < x < 2$$

iii).  $f(y/x)$

$$\text{WKT } f(y/x) = \frac{f(x, y)}{F(x)}$$

$$= \frac{\frac{1}{8} x(x-y)}{x^3/4}$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



$$= \frac{4/8 (x^2 - 2xy)}{x^3}$$

$$= \frac{1}{2} \frac{x(x-y)}{x^3}$$

$$F(y/x) = \frac{x-y}{2x^2}$$

2]. The joint probability density function

$$f(x, y) = \begin{cases} xy^2 + \frac{x^2}{8}, & 0 \leq x \leq 2 \text{ \& } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

i).  $P(x > 1 / y < \frac{1}{2})$     ii).  $P(y < \frac{1}{2} / x > 1)$

iii).  $P(x < y)$     iv).  $P(x + y \leq 1)$

Soln.

i).  $P(x > 1 / y < \frac{1}{2})$

$$= \frac{P(x > 1, y < \frac{1}{2})}{P(y < \frac{1}{2})} \rightarrow (1)$$

Now,  $P(x > 1, y < \frac{1}{2}) = \int_1^2 \int_0^{\frac{1}{2}} (xy^2 + \frac{x^2}{8}) dy dx$

$$= \int_1^2 \left[ \frac{xy^3}{3} + \frac{x^2}{8} y \right]_{y=0}^{\frac{1}{2}} dx$$

$$= \int_1^2 \left\{ \frac{x}{3} \left( \frac{1}{8} \right) + \frac{x^2}{8} \left( \frac{1}{2} \right) - 0 \right\} dx$$

$$= \int_1^2 \left( \frac{x}{24} + \frac{x^2}{16} \right) dx$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



$$\begin{aligned} &= \left( \frac{1}{24} \frac{x^2}{2} + \frac{1}{16} \frac{x^3}{3} \right) \Big|_0^2 \\ &= \left( \frac{4}{48} + \frac{8}{48} \right) - \left( \frac{1}{48} + \frac{1}{48} \right) \\ &= \frac{12}{48} - \frac{2}{48} \\ &= \frac{10}{48} \\ &= \frac{5}{24} \rightarrow (a) \end{aligned}$$

ii).  $P\left(y < \frac{1}{2} \mid x > 1\right)$   
 $= \frac{P(x > 1, y < \frac{1}{2})}{P(x > 1)}$

$$\begin{aligned} P\left(y < \frac{1}{2}\right) &= \int_0^2 \int_0^{\frac{1}{2}} f(x, y) dy dx \\ &= \int_0^2 \int_0^{\frac{1}{2}} \left( xy^2 + \frac{x^2}{8} \right) dy dx \\ &= \int_0^2 \left[ \frac{xy^3}{3} + \frac{x^2}{8} y \right]_{y=0}^{\frac{1}{2}} dx \\ &= \int_0^2 \left[ \frac{x}{3} \left( \frac{1}{8} \right) + \frac{x^2}{8} \left( \frac{1}{2} \right) \right] dx \\ &= \int_0^2 \left[ \frac{x}{24} + \frac{x^2}{16} \right] dx \\ &= \left[ \frac{x^2}{48} + \frac{x^3}{48} \right]_0^2 \\ &= \frac{4}{48} + \frac{8}{48} = \frac{12}{48} \end{aligned}$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



$$\begin{aligned} \text{(i)} \Rightarrow P(x > 1 \mid y < \frac{1}{2}) &= \frac{5}{24} \cdot \frac{4}{1} \\ &= \frac{5}{6} \end{aligned}$$

$$\begin{aligned} \text{ii). } P(y < \frac{1}{2} \mid x > 1) &= \frac{P(x > 1, y < \frac{1}{2})}{P(x > 1)} \rightarrow (3) \end{aligned}$$

Now

$$\begin{aligned} P(x > 1) &= \int_1^2 \int_0^1 f(x, y) dy dx \\ &= \int_1^2 \int_0^1 (xy^2 + \frac{x^2 y}{8}) dy dx \\ &= \int_1^2 \left[ \frac{xy^3}{3} + \frac{x^2 y^2}{8} \right]_{y=0}^1 dx \\ &= \int_1^2 \left[ \frac{x}{3} + \frac{x^2}{8} \right] dx \\ &= \left( \frac{x^2}{6} + \frac{x^3}{24} \right) \Big|_1^2 \\ &= \left( \frac{4}{6} + \frac{8}{24} \right) - \left( \frac{1}{6} + \frac{1}{24} \right) \\ &= \frac{16 + 8 - 4 - 1}{24} \\ &= \frac{19}{24} \end{aligned}$$

$$\text{(3)} \Rightarrow P(y < \frac{1}{2} \mid x > 1) = \frac{5}{24} \times \frac{24}{19} = \frac{5}{19}$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



iii)  $P(x < y)$

$$= \int_0^1 \int_0^y f(x, y) dx dy$$

$$= \int_0^1 \int_0^y \left( xy^2 + \frac{x^2}{8} \right) dx dy$$

$$= \int_0^1 \left[ \frac{x^2}{2} y^2 + \frac{x^3}{24} \right]_0^y dy$$

$$= \int_0^1 \left[ \frac{y^4}{2} + \frac{y^3}{24} \right] dy$$

$$= \left[ \frac{y^5}{10} + \frac{y^4}{96} \right]_0^1$$

$$\frac{24 \times 4}{96}$$

$$= \frac{1}{10} + \frac{1}{96}$$

$$= \frac{96+10}{960} = \frac{106}{960}$$

iv)  $P(x+y \leq 1)$

$$= \int_0^1 \int_0^{1-y} f(x, y) dx dy$$

$$= \int_0^1 \int_0^{1-y} \left( xy^2 + \frac{x^2}{8} \right) dx dy$$

$$= \int_0^1 \left[ \frac{x^2}{2} y^2 + \frac{x^3}{24} \right]_{x=0}^{1-y} dy$$

$$= \int_0^1 \left[ \frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24} \right] dy$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



$$\begin{aligned}
&= \int_0^1 \left[ \frac{(1+y^2-2y)y^2}{2} + \frac{(1-y)(1-y)^2}{24} \right] dy \\
&= \int_0^1 \left[ \frac{y^2+y^4-2y^3}{2} + \frac{(1-y)(1+y^2-2y)}{24} \right] dy \\
&= \int_0^1 \left[ \frac{y^2+y^4-2y^3}{2} + \frac{1+y^2-2y-y-y^3+2y^2}{24} \right] dy \\
&= \frac{1}{24} \int_0^1 [12(y^2+y^4-2y^3) + (1+3y^2-3y-y^3)] dy \\
&= \frac{1}{24} \int_0^1 [12y^2+12y^4-24y^3+1+3y^2-3y-y^3] dy \\
&= \frac{1}{24} \int_0^1 [12y^4-25y^3+15y^2-3y+1] dy \\
&= \frac{1}{24} \left[ \frac{12y^5}{5} - \frac{25y^4}{4} + \frac{15y^3}{3} - \frac{3y^2}{2} + y \right]_{y=0}^1 \\
&= \frac{1}{24} \left[ \left( \frac{12}{5} - \frac{25}{4} + \frac{15}{3} - \frac{3}{2} + 1 \right) - 0 \right] \quad \begin{matrix} 2 \\ \hline 5, 4, 3, 2 \\ \hline 5, 2, 3, 1 \end{matrix} \\
&= \frac{1}{24} \left[ \frac{144-375+300-90+60}{60} \right] \\
&= \frac{39}{1440} \\
&= 0.027
\end{aligned}$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



3) The joint PDF of the RV is given by,  
 $f(x, y) = kxy e^{-(x^2+y^2)}$ ,  $x > 0, y > 0$ .

find i).  $k$  ii). check  $x$  &  $y$  are independent.

Soln.

$$i). \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$$

$$\int_0^{\infty} \int_0^{\infty} kxy e^{-(x^2+y^2)} dy dx = 1$$

$$k \int_0^{\infty} \int_0^{\infty} xy e^{-x^2} e^{-y^2} dy dx = 1$$

$$\text{Take } x^2 = s \quad \left| \quad y^2 = t \right.$$
$$ds = 2x dx \quad \left| \quad 2y dy = dt \right.$$
$$\frac{ds}{2} = x dx \quad \left| \quad y dy = \frac{dt}{2} \right.$$

Now,

$$k \int_0^{\infty} \int_0^{\infty} e^{-s} e^{-t} \frac{dt}{2} \frac{ds}{2} = 1$$

$$\frac{k}{4} \int_0^{\infty} \int_0^{\infty} e^{-s} e^{-t} dt ds = 1$$

$$\frac{k}{4} \int_0^{\infty} e^{-s} \left[ \frac{e^{-t}}{-1} \right]_0^{\infty} ds = 1$$

$$-\frac{k}{4} \int_0^{\infty} e^{-s} [0 - 1] ds = 1$$

$$\frac{k}{4} \left( \frac{e^{-s}}{-1} \right)_0^{\infty} = 1$$





# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



$$-\frac{k}{4}(0-1) = 1$$

$$\frac{k}{4} = 1$$

$$\boxed{k=4}$$

ii).  $x$  &  $y$  are independent.

To prove  $F(x, y) = F(x) \cdot F(y)$

Now,

$$\begin{aligned} F(x) &= \int_0^{\infty} F(x, y) dy \\ &= \int_0^{\infty} 4xy e^{-(x^2+y^2)} dy \\ &= 4x e^{-x^2} \int_0^{\infty} y e^{-y^2} dy \end{aligned}$$

Take  $y^2 = t$

$$2y dy = dt$$

$$y dy = \frac{dt}{2}$$

$$= 4x e^{-x^2} \int_0^{\infty} e^{-t} \frac{dt}{2}$$

$$= 2x e^{-x^2} \left[ \frac{e^{-t}}{-1} \right]_0^{\infty}$$

$$= -2x e^{-x^2} (0-1)$$

$$F(x) = 2x e^{-x^2}, \quad x > 0$$

$$F(y) = \int_0^{\infty} F(x, y) dx$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



$$= \int_0^{\infty} 4xy e^{-(x^2+y^2)} dx$$

$$= 4y e^{-y^2} \int_0^{\infty} x e^{-x^2} dx$$

Take  $x^2 = s$

$$2x dx = ds$$

$$x dx = \frac{ds}{2}$$

$$= 4y e^{-y^2} \int_0^{\infty} e^{-s} \frac{ds}{2}$$

$$= 2y e^{-y^2} \left[ \frac{e^{-s}}{-1} \right]_0^{\infty}$$

$$= -2y e^{-y^2} (0-1)$$

$$f(y) = 2y e^{-y^2}, \quad y > 0$$

$$\therefore f(x) \cdot f(y) = 2x e^{-x^2} \cdot 2y e^{-y^2}$$

$$= 4xy e^{-(x^2+y^2)}$$

$$= f(x, y)$$

$\therefore x$  &  $y$  are independent.

AJ. IF the joint density function of  $x$  &  $y$  is given by,

$$f(x, y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

To prove  $x$  &  $y$  are independent.

Soln.

$$\text{WHT } f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



$$= \frac{\partial^2}{\partial x \partial y} [1 - e^{-x} - e^{-y} + e^{-(x+y)}]$$

$$= \frac{\partial}{\partial x} [-e^{-y}(-1) + e^{-x} e^{-y}(-1)]$$

$$= \frac{\partial}{\partial x} [e^{-y} - e^{-x} e^{-y}]$$

$$= 0 - e^{-y} e^{-x}(-1)$$

$$= e^{-x} e^{-y}$$

$$F(x, y) = e^{-(x+y)}$$

To prove:

$$F(x, y) = f(x) \cdot f(y)$$

Now,

$$f(x) = \int_0^{\infty} e^{-(x+y)} dy$$

$$= e^{-x} \int_0^{\infty} e^{-y} dy$$

$$= e^{-x} \left( \frac{e^{-y}}{-1} \right)_0^{\infty}$$

$$= -e^{-x} [0 - 1]$$

$$f(x) = e^{-x}$$

$$f(y) = \int_0^{\infty} e^{-(x+y)} dx$$

$$= \int_0^{\infty} e^{-x} e^{-y} dx$$

$$= e^{-y} \left( \frac{e^{-x}}{-1} \right)_0^{\infty}$$

$$= -e^{-y} (0 - 1)$$



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore - 641 035

DEPARTMENT OF MATHEMATICS

Marginal distribution, Conditional distribution



$$F(y) = e^{-y}$$

$$\begin{aligned} F(x) \cdot F(y) &= e^{-x} \cdot e^{-y} \\ &= e^{-(x+y)} \end{aligned}$$

$$= F(x, y)$$

$\therefore$   $x$  and  $y$  are independent.