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Transformation of two-dimensional random variable :

Step 1 : Find the joint density function of (X, Y) if it is not given.

Step 2 : Consider the new random variables U & V and from this find x and y.

Find $$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Step 3 : Find the pdf of (u, v).

i.e., $f_{UV}(u,v) = f_{XY}(x,y)\,|J|$

Step 4 : Find the values of $f_U(u)$ and $f_V(v)$ using the method of finding the marginal densities.

Step 5 : Change the domain values in terms of u, v using the given relation.

PROBLEMS :

1. If X and Y are independent random variables having density functions

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x < 0 \end{cases} \quad \text{and} \quad g(y) = \begin{cases} 3e^{-3y}, & y > 0 \\ 0, & y < 0 \end{cases}$$

find the density function of their sum $U = X + Y$

Solution :

Step 1 :

$$f(x) = \begin{cases} 2e^{-2x} & , x > 0 \\ 0 & , x < 0 \end{cases}$$

$$g(y) = \begin{cases} 3e^{-3y} & , y > 0 \\ 0 & , y < 0 \end{cases}$$

Since x and y are independent,

$$f(x,y) = f(x) \cdot f(y)$$

$$= 2e^{-2x} \cdot 3e^{-3y}$$

$$\boxed{f(x,y) = 6e^{-(2x+3y)}} \quad , x > 0 , y > 0$$

Step 2 :

$$U = X + Y$$

Let $V = X$

i.e., $u = x + y \quad ; \quad v = x$

$$u = v + y \quad ; \quad x = v$$

$$y = u - v \quad ; \quad x = v$$

$$x = v \Rightarrow \frac{\partial x}{\partial u} = 0 \quad ; \quad \frac{\partial x}{\partial v} = 1$$

$$y = u - v \Rightarrow \frac{\partial y}{\partial u} = 1 \quad ; \quad \frac{\partial y}{\partial v} = -1$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = |0 - 1| = 1$$

$$|J| = 1$$

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Step 3:

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$$
$$= 6e^{-(2x+3y)}, \quad x>0;\ y>0$$
$$= 6e^{-(2v+3(u-v))}$$
$$= 6e^{-(2v+3u-3v)}$$
$$= 6e^{-(3u-v)}$$
$$\boxed{f_{UV}(u,v) = 6e^{v-3u}}$$

Step 4:

Given: $x>0$ & $y>0$

i.e., $v>0$ & $u-v>0$

$u>v$

$\therefore\ u>v>0 \Rightarrow 0<v<u$

i.e., $u>0$ & $0<v<u$

Step 5:

Hence the pdf of U & V is,

$$f_{UV}(u,v) = \begin{cases} 6e^{v-3u}, & u>0,\ 0<v<u \\ 0, & \text{otherwise} \end{cases}$$

Step 6: The density function of U is,

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv$$
$$= \int_0^u 6e^{v-3u}\,dv$$

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$$f_U(u) = 6e^{-3u} \int_0^u e^v \, dv$$

$$= 6e^{-3u} \left[e^v \right]_0^u$$

$$f_U(u) = 6e^{-3u} (e^u - 1), \quad u > 0$$

$$\therefore \boxed{f_U(u) = \begin{cases} 6e^{-3u}(e^u - 1), & u > 0 \\ 0, & \text{otherwise} \end{cases}}$$

(2) Let (X, Y) be a two-dimensional random variable having the joint density

$$f_{XY}(x,y) = \begin{cases} 4xy\, e^{-(x^2+y^2)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the density function of $U = \sqrt{X^2+Y^2}$

Solution :

Step 1 :

$$f_{XY}(x,y) = \begin{cases} 4xy\, e^{-(x^2+y^2)}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Step 2 :

$$U = \sqrt{X^2+Y^2}$$

Let $V = X$

i.e., $u^2 = x^2 + y^2, \quad v = x$

$x = v \quad ; \quad u^2 = v^2 + y^2$

$$y^2 = u^2 - v^2$$

$$y = \sqrt{u^2 - v^2}$$

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