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Given : $f(x,y) = \frac{1}{3}(x+y)$, $0 \le x \le 1$, $0 \le y \le 2$.

(i) To find Correlation Coefficient:

$$r(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} \to (1)$$

First let us find $E(X)$, $E(Y)$, $E(XY)$, $Var(X)$ and $Var(Y)$.

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy$$

$$= \int_0^2 \int_0^1 x \cdot \frac{1}{3}(x+y)\, dx\, dy$$

$$= \frac{1}{3}\int_0^2 \int_0^1 (x^2 + xy)\, dx\, dy$$

$$= \frac{1}{3}\int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^1 dy$$

$$= \frac{1}{3}\int_0^2 \left[\frac{1}{3} + \frac{y}{2}\right] dy$$

$$= \frac{1}{3}\left[\frac{y}{3} + \frac{y^2}{4}\right]_0^2$$

$$= \frac{1}{3}\left[\frac{2}{3} + \frac{4}{4}\right] = \frac{1}{3}\left[\frac{2}{3} + 1\right] = \frac{2+3}{9}$$

$$\boxed{E(X) = \frac{5}{9}}$$

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$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy$$

$$= \int_0^2\int_0^1 y \cdot \frac{1}{3}(x+y)\, dx\, dy$$

$$= \frac{1}{3}\int_0^2\int_0^1 \left[xy + y^2\right] dx\, dy$$

$$= \frac{1}{3}\int_0^2 \left[\frac{x^2 y}{2} + y^2 x\right]_0^1 dy$$

$$= \frac{1}{3}\int_0^2 \left[\frac{y}{2} + y^2\right] dy$$

$$= \frac{1}{3}\left[\frac{y^2}{4} + \frac{y^3}{3}\right]_0^2 = \frac{1}{3}\left[\frac{4}{4} + \frac{8}{3}\right]$$

$$= \frac{3+8}{9} = \frac{11}{9}$$

$$\boxed{E(Y) = \frac{11}{9}}$$

$$E(X^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)\, dx\, dy$$

$$= \int_0^2\int_0^1 x^2 \cdot \frac{1}{3}(x+y)\, dx\, dy$$

$$= \frac{1}{3}\int_0^2\int_0^1 (x^3 + x^2 y)\, dx\, dy$$

$$= \frac{1}{3}\int_0^2 \left[\frac{x^4}{4} + \frac{x^3 y}{3}\right]_0^1 dy$$

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$$E(x^2) = \frac{1}{3}\int_0^2 \left[\frac{1}{4}+\frac{y}{3}\right] dy$$

$$= \frac{1}{3}\left[\frac{y}{4}+\frac{y^2}{6}\right]_0^2$$

$$= \frac{1}{3}\left[\frac{2}{4}+\frac{4}{6}\right] = \frac{1}{3}\left[\frac{1}{2}+\frac{2}{3}\right]$$

$$= \frac{1}{3}\left[\frac{3+4}{6}\right] = \frac{7}{18}$$

$$\boxed{E(x^2) = \frac{7}{18}}$$

$$E(y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)\,dx\,dy$$

$$= \int_0^2\int_0^1 y^2 \frac{1}{3}(x+y)\,dx\,dy$$

$$= \frac{1}{3}\int_0^2\int_0^1 (y^2x+y^3)\,dx\,dy$$

$$= \frac{1}{3}\int_0^2 \left[\frac{y^2x^2}{2}+y^3x\right]_0^1 dy$$

$$= \frac{1}{3}\int_0^2 \left[\frac{y^2}{2}+y^3\right] dy$$

$$= \frac{1}{3}\left[\frac{y^3}{6}+\frac{y^4}{4}\right]_0^2 = \frac{1}{3}\left[\frac{8}{6}+\frac{16}{4}\right]$$

$$= \frac{1}{3}\left[\frac{4}{3}+4\right] = \frac{1}{3}\left[\frac{4+12}{3}\right] = \frac{16}{9}$$

$$\boxed{E(y^2) = \frac{16}{9}}$$

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{7}{18} - \frac{25}{81}$$
$$\boxed{Var(X) = \frac{13}{162}} \Rightarrow \sigma_x^2 = \frac{13}{162} \Rightarrow \boxed{\sigma_X = \sqrt{\frac{13}{162}}}$$
$$Var(Y) = E(Y^2) - [E(Y)]^2$$
$$= \frac{16}{9} - \left(\frac{11}{9}\right)^2$$
$$\boxed{Var(Y) = \frac{23}{81}} \Rightarrow \sigma_y^2 = \frac{23}{81} \Rightarrow \boxed{\sigma_y = \sqrt{\frac{23}{81}}}$$
$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$
$$= \int_0^2\int_0^1 xy\, \frac{1}{3}(x+y)\, dx\, dy$$
$$= \frac{1}{3}\int_0^2\int_0^1 (x^2y + xy^2)\, dx\, dy$$
$$= \frac{1}{3}\int_0^2 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy$$
$$= \frac{1}{3}\int_0^2 \left[\frac{y}{3} + \frac{y^2}{2}\right] dy$$
$$= \frac{1}{3}\left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^2 = \frac{1}{18}(4+8) = \frac{12}{18}$$
$$\boxed{E(XY) = \frac{2}{3}}$$

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$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{2}{3} - \frac{5}{9} \cdot \frac{11}{9}$$

$$= \frac{2}{3} - \frac{55}{81} = \frac{-1}{81}$$

$$\boxed{Cov(X,Y) = \frac{-1}{81}}$$

$$\therefore \quad r(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{-1/81}{\sqrt{\frac{13}{162} \times \frac{23}{81}}} = -\sqrt{\frac{2}{299}}$$

$$\boxed{r = -\sqrt{\frac{2}{299}} = -0.0818}$$

(ii) Two lines of regression :

Equation of the line of regression of X on Y is,

$$x - \bar{x} = r \cdot \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$x - \frac{5}{9} = (-0.0818) \frac{\sqrt{13/162}}{\sqrt{23/81}} \left(y - \frac{11}{9}\right)$$

$$\boxed{x - \frac{5}{9} = (-0.0435)\left(y - \frac{11}{9}\right)}$$

Equation of the line of regression of Y on X is,

$$y - \bar{y} = r \cdot \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$