POISSON DISTRIBUTION

Definition: A random variable X is said to follow Poisson distribution if it assumes only non-negative values and its probability mass function is given by,

$$P(X=x) = p(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!}, & x = 0,1,2,\dots\infty \\ 0, & \text{otherwise.} \end{cases}$$

Moment Generating function:

$$M_X(t) = E(e^{tx})$$
$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$
$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} e^{tx}$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$
$$= e^{-\lambda}\left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right]$$
$$= e^{-\lambda} e^{\lambda e^t} \qquad \left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$
$$\boxed{M_X(t) = e^{\lambda(e^t - 1)}}$$

Mean and Variance:

$$\mu_1' = E(X)$$
$$= \sum_{x=0}^{\infty} x \cdot p(x)$$
$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$
$$= e^{-\lambda}\left[0 + 1\cdot\frac{\lambda}{1!} + 2\cdot\frac{\lambda^2}{2!} + 3\frac{\lambda^3}{3!} + \cdots\right]$$

$$\mu_1' = \lambda e^{-\lambda}\left[1+\frac{\lambda}{1!}+\frac{\lambda}{2!}+\frac{\lambda^2}{3!}+\cdots\right]$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{\mu_1' = \text{Mean} = \lambda}$$

$$\mu_2' = E(X^2)$$

$$= \sum_{x=0}^{\infty} x^2 \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x]\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{\cancel{x}(\cancel{x-1})e^{-\lambda}\lambda^{x-2}\lambda^2}{\cancel{x}(\cancel{x-1})(x-2)\cdots 1} + \lambda \quad (\because \mu_1' = \lambda)$$

$$= e^{-\lambda}\lambda^2 \sum_{x=0}^{\infty}\frac{\lambda^{x-2}}{(x-2)(x-3)\cdots 1} + \lambda$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2\left[1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\cdots\right]+\lambda$$

$$= +e^{-\lambda}\lambda^2 e^{\lambda} + \lambda$$

$$\boxed{\mu_2' = \lambda^2+\lambda}$$

$$\text{Variance} = \mu_2' - \mu_1'^2$$

$$= \lambda^2+\lambda-\lambda^2 = \lambda$$

$$\boxed{\text{Variance} = \lambda}$$

(or)

Mean:

$$M_X(t) = e^{\lambda(e^t-1)} = e^{\lambda e^t} \cdot e^{-\lambda}$$

$$M_X'(t) = \lambda e^t e^{\lambda e^t} e^{-\lambda}$$

$$M_X'(0) = \lambda \cdot e^{\lambda} \cdot e^{-\lambda}$$

$$M_X'(0) = \lambda$$

$$\boxed{\text{Mean} = E(X) = M_X'(0) = \lambda}$$

$$M_X''(t) = E(X^2) = (\lambda e^t)^2 e^{\lambda e^t} e^{-\lambda} + \lambda e^{\lambda e^t} e^{-\lambda}$$

$$M_X''(0) = \lambda^2 e^{\lambda} e^{-\lambda} + \lambda e^{\lambda} e^{-\lambda}$$

$$E(X^2) = M_X''(0) = \lambda^2 + \lambda$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$= \lambda^2 - \lambda^2 + \lambda$$

$$\boxed{\text{Variance} = \lambda}$$

Prove that poisson distribution is the limiting case of binomial distribution.

Suppose in a binomial distribution,

1. The number of trials is indefinitely large i.e., $n \to \infty$
2. p is very small i.e., $p \to 0$
3. $np = \lambda$ is finite.

Now $P(X = x) = nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots n$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}\right]$$

Taking limit as $n \to \infty$,

$$\lim_{n\to\infty} p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \text{ for } x = 0, 1, 2, \ldots$$

$$\left(\because \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}\right)$$

which is the p.m.f of the poisson distribution.

Problems :

① If X is a poisson Variate,

$P(X=2) = 9P(X=4) + 90\, P(X=6)$,

find (i) mean of X (ii) Variance of X ~~(iii) $P(X \geq 2)$~~ ~~(iv) $E(X^2)$~~.

Solution:

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,1,2,\ldots$$

Given: $P(X=2) = 9P(X=4) + 90\,P(X=6)$

i.e., $$\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{1}{2}(\cancel{e^{-\lambda}}\cancel{\lambda^2}) = \cancel{e^{-\lambda}}\cancel{\lambda^2}\left(\frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}\right)$$

$$\frac{1}{2} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$$

$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$$4 = \lambda^4 + 3\lambda^2$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$\lambda^2 = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = 1 \text{ or } -4$$

$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$ $\quad\lambda = -1$ is also not possible.

$\lambda^2 = -4$ is not possible.

$\therefore$ $\underline{\text{Mean} = \lambda = 1}$

$\underline{\text{Variance} = \lambda = 1}$

(2) The number of monthly breakdown of a computer is a random variable having a poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (i) with only one breakdown (ii) without a breakdown (iii) with atleast one breakdown.

Solution :

Let X denotes the number of breakdowns in a month.

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Given : Mean $= \lambda = 1.8$.

$$\therefore P(X=x) = \frac{e^{-1.8}(1.8)^{x}}{x!}$$

(i) P(with only one breakdown)

$$= P(X=1)$$

$$= \frac{e^{-1.8}(1.8)^{1}}{1!}$$

$$= 0.2975$$

(ii) P(without a break down) $= P(X=0)$

$$= \frac{e^{-1.8}(1.8)^0}{0!} = e^{-1.8}$$

$$= 0.1653$$

(iii) P(Atleast one breakdown) $= P(X \geq 1)$

$$= 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - 0.1653$$

$$= 0.8347.$$

③ In a certain factory turning razar blades there is a small chance of 1/500 for any blade to be defective: The blades are in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (i) no defective (ii) one defective (iii) 2 defective blades respectively in a consignment of 10,000 packets.

Solution :

Let X denote the number of defective blades.

Given : $p = \frac{1}{500}$, $n = 10$

$N = 10,000$

$$\text{Mean} = \lambda = np = 10 \times \frac{1}{500} = 0.02$$

$$\lambda = 0.02$$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \frac{e^{-0.02}(0.02)^x}{x!}$$

(i) $P(\text{no defective blades}) = P(X=0)$

$$= \frac{e^{-0.02}(0.02)^0}{0!} = e^{-0.02}$$

$$= 0.9802$$

$\therefore$ Total number of packets containing no defective blades in 10,000 packets

$$= N \times P(\text{no defective})$$

$$= 10{,}000 \times 0.9802$$

$$= \underline{9802 \text{ packets}}.$$

(ii) $P(\text{one defective blade}) = P(X=1)$

$$= \frac{e^{-0.02}(0.02)^1}{1!}$$

$$= 0.01960$$

Number of packets containing one defective blades

$$= N \times P(\text{one defective})$$

$$= 10{,}000 \times 0.01960$$

$$= \underline{196 \text{ packets}}.$$

(iii) $P(\text{two defective blades}) = P(X=2)$

$$= \frac{e^{-0.02}(0.02)^2}{2!}$$

$$= 0.000196$$

Number of packets containing two defective blades

$$= N \times P(\text{two defectives})$$

$$= 10{,}000 \times 0.000196$$

$$\simeq \underline{2 \text{ packets}}.$$

HW (4) The average number of traffic accidents on a certain section of a highway is two per week. Assume that the number of accidents follow a poisson distribution. Find the probability of (i) no accident in a week (ii) atmost two accidents in a 2 week period

Solution:

No of accidents per week, $\lambda = 2$

$$P(X = x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= \frac{e^{-2} 2^{x}}{x!}$$

(i) $P(\text{no accidents}) = P(X=0) = \frac{e^{-2} 2^{0}}{0!} = e^{-2}$

$= 0.1353$.

(ii) During a 2 week period the average number of accidents $= 2 + 2 = 4$. Here $\lambda = 4$

$\therefore P(\text{atmost } \cancel{2}\ 4 \text{ accidents during 2-week period}) = P(X<4)$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{e^{-\cancel{2}} \cancel{2}^{0}_{4}}{0!} + \frac{e^{-2} \cancel{2}^{1}_{4}}{1!} + \frac{e^{-2} \cancel{2}^{2}_{4}}{2!} + \frac{e^{-\cancel{2}} \cancel{2}^{3}_{4}}{3!}$$

$$= \frac{e^{-4} 4^{0}}{0!} + \frac{e^{-4} 4^{1}}{1!} + \frac{e^{-4} 4^{2}}{2!} + \frac{e^{-4} 4^{3}}{3!}$$

$$= e^{-4} \left[1 + 4 + \frac{16}{2} + \frac{64}{6}\right]$$

$$= e^{-4} (23.67)$$

$$= 0.4335.$$

(5) If X and Y are independent random variables. Show that the conditional distribution of X given X+Y is a binomial distribution.

Solution:

Let X and Y be independent random variables with parameters λ_1 & λ_2.

$$P(X = r / X+Y = n) = \frac{P(X = r \cap X+Y = n)}{P(X+Y = n)}$$

$$= \frac{P(X = r \cap Y = n-r)}{P(X+Y = n)}$$

$$= \frac{P(X = r) \cdot P(Y = n-r)}{P(X+Y = n)}$$

$$= \frac{\dfrac{e^{-\lambda_1} \lambda_1^r}{r!} \dfrac{e^{-\lambda_2} \cdot \lambda_2^{n-r}}{(n-r)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{r!\,(n-r)!} \frac{\lambda_1^r \; \lambda_2^{n-r}}{(\lambda_1+\lambda_2)^{n-r+r}}$$

$$= nC_r \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-r}$$

$$= nC_r \, p^r q^{n-r} \quad \text{where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$q = \frac{\lambda_2}{\lambda_1+\lambda_2}$$