CORRELATION CO-EFFICIENTS :

Let x and y be given random variables.
The correlation coefficient is denoted by γ_{xy} or ρ_{xy} and defined as

$$\rho_{xy} = \gamma_{xy} = \frac{\text{cov}(x,y)}{\sigma_x \cdot \sigma_y} = \frac{\text{cov}(x,y)}{\sqrt{\text{var}(x)}\sqrt{\text{var}(y)}}$$

Discrete Random Variable	Continuous Random Variable
$\text{Cov}(x,y) = \frac{\sum xy}{n} - \frac{\sum x}{n} \cdot \frac{\sum y}{n}$	$\text{Cov}(x,y) = E(xy) - E(x) \cdot E(y)$
$\sigma_x = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$	$\sigma_x = \sqrt{E(x^2) - [E(x)]^2}$
$\sigma_y = \sqrt{\frac{\sum y^2}{n} - \left(\frac{\sum y}{n}\right)^2}$	$\sigma_y = \sqrt{E(y^2) - [E(y)]^2}$

Note: $\text{Cov}(x,y) = 0$, If x & y are independent (uncorrelated)

1) Find the correlation coefficient between the marks obtained by 10 students in physics & chemistry.

x (phy) :	65	45	40	55	60	50	80	30	70	65
y (chem) :	60	60	55	70	80	40	85	50	70	80

Soln:

x (phy)	y (chem)	x^2	y^2	xy
65	60	4225	3600	3900
45	60	2025	3600	2700
40	55	1600	3025	2200
55	70	3025	4900	3850
60	80	3600	6400	4800
50	40	2500	1600	2000
80	85	6400	7225	6800
30	50	900	2500	1500
70	70	4900	4900	4900
65	80	4225	6400	5200

$\Sigma x : 560 \quad \Sigma y : 650 \quad \Sigma x^2 : 33400 \quad \Sigma y^2 : 44150 \quad \Sigma xy : 37850$

$\frac{\Sigma x}{n} : 56 \quad \frac{\Sigma y}{n} = 65 \quad \frac{\Sigma x^2}{n} = 3340 \quad \frac{\Sigma y^2}{n} : 4415 \quad \frac{\Sigma xy}{n} : 3785$

Now $\text{cov}(x,y) = \frac{\Sigma xy}{n} - \frac{\Sigma x}{n} \cdot \frac{\Sigma y}{n}$

$= 3785 - 56 \times 65 = 145$

$$\sigma_x = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{3340 - (56)^2} = \sqrt{204} = 14.28$$

$$\sigma_y = \sqrt{\frac{\Sigma y^2}{n} - \left(\frac{\Sigma y}{n}\right)^2} = \sqrt{4415 - (65)^2} = \sqrt{190} = 13.78$$

$$\therefore \rho_{xy} = r_{xy} = \frac{Cov(x,y)}{\sigma_x \cdot \sigma_y}$$

$$= \frac{145}{14.28 \times 13.78}$$

$$= 0.7368$$

JPDF of Random Variable x & y is $f(x,y) = \begin{cases} 2-x-y, & 0 \le x \le 1, \; 0 \le y \le 1 \\ 0, & \text{otherwi} \end{cases}$ find correlation coefficient between x & y.

Soln:

WKT $\rho_{xy} = r_{xy} = \dfrac{Cov(x,y)}{\sigma_x \cdot \sigma_y}$

where $Cov(x,y) = E(xy) - E(x) \cdot E(y)$

$$\sigma_x = \sqrt{E(x^2) - (E(x))^2}$$

$$\sigma_y = \sqrt{E(y^2) - [E(y)]^2}$$

Now $E(x) = \int_0^1 \int_0^1 x(2-x-y)\,dx\,dy$

$$= \int_0^1 \left. \frac{2x^2}{2} - \frac{x^3}{3} - \frac{yx^2}{2} \right]_0^1 dy$$

$$= \int_0^1 \left(1 - \frac{1}{3} - \frac{y}{2}\right) dy$$

$$= \left. y - \frac{y}{3} - \frac{y^2}{4} \right]_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = 5/12$$

$\therefore E(x) = 5/12$

$E(x^2) = \int_0^1 \int_0^1 x^2(2-x-y)\,dx\,dy$

$$= \int_0^1 \left. \frac{2x^3}{3} - \frac{x^4}{4} - \frac{yx^3}{3} \right]_0^1 dy$$

$$= \int_0^1 \left(\frac{2}{3} - \frac{1}{4} - \frac{y}{3}\right) dy$$

$$= \left. \frac{2y}{3} - \frac{1y}{4} - \frac{y^2}{6} \right]_0^1 = \frac{2}{3} - \frac{1}{4} - \frac{1}{6} = 3/12 = 1/4$$

$\therefore E(x^2) = 1/4$

$$E(y) = \int_0^1\int_0^1 y\,(2-x-y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{2y^2}{2} - \frac{xy^2}{2} - \frac{y^3}{3}\right]_0^1 dx$$

$$= \int_0^1 \left(1 - \frac{x}{2} - \frac{1}{3}\right) dx$$

$$= \left[x - \frac{x^2}{4} - \frac{x}{3}\right]_0^1 = 1 - \frac{1}{4} - \frac{1}{3} = 5/12$$

$$\therefore E(y) = 5/12$$

$$E(y^2) = \int_0^1\int_0^1 y^2\,(2-x-y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{2y^3}{3} - \frac{xy^3}{3} - \frac{y^4}{4}\right]_0^1 dx$$

$$= \int_0^1 \left[\frac{2}{3} - \frac{x}{3} - \frac{1}{4}\right] dx$$

$$= \left[\frac{2}{3}x - \frac{x^2}{6} - \frac{x}{4}\right]_0^1 = \frac{2}{3} - \frac{1}{6} - \frac{1}{4} = 3/12 = 1/4$$

$$\therefore E(y^2) = 1/4$$

$$E(xy) = \int_0^1 \int_0^1 xy\,(2-x-y)\,dx\,dy$$

$$= \int_0^1 \left[2y\frac{x^2}{2} - y\frac{x^3}{3} - y^2\frac{x^2}{2} \right]_0^1 dy$$

$$= \int_0^1 \left[y - \frac{y}{3} - \frac{y^2}{2} \right] dy$$

$$= \left[\frac{y^2}{2} - \frac{y^2}{6} - \frac{y^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{6} - \frac{1}{6} = 1/6$$

$\therefore E(xy) = 1/6$

Now $\text{cov}(x,y) = E(xy) - E(x)\cdot E(y)$

$$= \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12}$$

$$= -\frac{1}{144}$$

$$\sigma_x = \sqrt{E(x^2) - (E(x))^2}$$

$$= \sqrt{1/4 - (5/12)^2}$$

$$= \sqrt{\frac{11}{144}}$$

$$\sigma_y = \sqrt{E(y^2) - (E(y))^2}$$

$$= \sqrt{1/4 - (5/12)^2} = \sqrt{\frac{11}{144}}$$

$$\therefore \rho_{xy} = \gamma_{xy} = \frac{\text{Cov}(x,y)}{\sigma_x \cdot \sigma_y} = \frac{-1/144}{\sqrt{11/144}\cdot\sqrt{11/144}} = \frac{-1/144}{11/144} = -1/11$$

$$= -0.09$$