CONTINUOUS RANDOM VARIABLE:

(1) If the joint pdf of a 2D R.V (X, Y) is given by

$$f(x,y) = k(6-x-y), \quad 0<x<2, \; 2<y<4$$
$$= 0, \quad \text{otherwise}.$$

(i) Find the value of k.

(ii) Find MDF of X and Y.

(iii) $P(X<1, Y<3)$

(iv) $P(X+Y<3)$ and

(v) $P(X<1 / Y<3)$

Soln:

To find k: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

$$\Rightarrow \int_0^2 \int_2^4 k(6-x-y)\,dy\,dx = 1$$

$$\Rightarrow k\int_0^2 \left[6y - xy - \frac{y^2}{2}\right]_2^4 dx = 1$$

$$\Rightarrow k\int_0^2 \left\{24 - 4x - \frac{16}{2} - \left[12 - 2x - \frac{4}{2}\right]\right\} dx = 1$$

$$\Rightarrow k\int_0^2 (6-2x)\,dx = 1$$

$$\Rightarrow k\left[6x-\frac{2x^2}{2}\right]_0^2 = 1$$

$$\Rightarrow k\,[12-4] = 1$$

$$\Rightarrow k = 1/8$$

$$\therefore f(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4 \\ 0, & \text{otherwise} \end{cases}$$

(ii) MDF of x and y:

$$\int_{-\infty}^{\infty} f(x,y)\,dy = \int_2^4 \frac{1}{8}(6-x-y)\,dy$$

$$= \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4$$

$$= \frac{1}{8}\left[24 - 4x - \frac{16}{2} - \left(12 - 2x - \frac{4}{2}\right)\right]$$

$$= \frac{1}{8}[6-2x]$$

$$\therefore f(x) = \frac{1}{8}(6-2x),\ 0<x<2$$

Now MDF of y:

$$\int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{1}{8}(6-x-y)\,dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - yx\right]_0^2$$

$$= \frac{1}{8}\left[12 - \frac{4}{2} - 2y\right]$$

$$= \frac{1}{8}\left[10 - 2y\right]$$

$$\therefore f(y) = \frac{1}{8}(10-2y),\quad 2<y<4$$

(iii) $p(x<1, y<3) = \int_0^1\int_2^3 f(x,y)\,dy\,dx$

$$= \int_0^1\int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^1 \left.6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^1 \left\{18 - 3x - \frac{9}{2} - \left(12 - 2x - \frac{4}{2}\right)\right\} dx$$

$$= \frac{1}{8}\int_0^1 \left[\frac{7}{2} - x\right] dx$$

$$= \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{7}{2} - \frac{1}{2}\right] = \frac{3}{8}$$

(iv) $$P(X+Y<3) = \int_2^3 \int_{x=0}^{x=3-y} f(x,y)\,dx\,dy \qquad x+y=3,\quad x=3-y$$

$$= \int_2^3 \int_{x=0}^{x=3-y} \frac{1}{8}(6-x-y)\,dx\,dy$$

$$= \frac{1}{8}\int_2^3 \left[6x - \frac{x^2}{2} - yx\right]_0^{3-y} dy$$

$$= \frac{1}{8}\int_2^3 \left[6(3-y) - \frac{(3-y)^2}{2} - y(3-y)\right] dy$$

$$= \frac{1}{8}\int_2^3 \left(18 - 6y - \frac{(9+y^2-6y)}{2} - 3y + y^2\right) dy$$

$$= \frac{1}{8}\int_2^3 \left(18 - 9y + y^2 - \frac{9}{2} - \frac{y^2}{2} + 3y\right) dy$$

$$= \frac{1}{8}\int_2^3 \left(\frac{27}{2} - 6y + \frac{y^2}{2}\right) dy$$

$$= \frac{1}{8}\left[\frac{27}{2}y - \frac{6y^2}{2} + \frac{y^3}{6}\right]_2^3$$

$$= \frac{1}{8}\left[\frac{81}{2} - 27 + \frac{9}{2} - \left(\frac{54}{2} - 12 + \frac{4}{3}\right)\right]$$

$$= \frac{1}{8}\left[\frac{81}{2} - 27 + \frac{9}{2} - 27 + 12 - \frac{4}{3}\right]$$

$$= \frac{1}{8}\left[\frac{90}{2} - 54 + 12 - \frac{4}{3}\right]$$

$$= \frac{1}{8}\left[57 - 54 + \frac{4}{3}\right]$$

$$= \frac{5}{24}$$

(v) $p(x<1 / y<3) = \dfrac{p(x<1, y<3)}{p(y<3)}$

WKT $p(x<1, y<3) = 3/8$

Now $p(y<3) = \int_0^2\int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$

$$= \int_0^2 \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^2 \left\{18 - 3x - \frac{9}{2} - \left[12 - 2x - \frac{4}{2}\right]\right\} dx$$

$$= \frac{1}{8} \int_0^2 \left(6 - x - \frac{5}{2}\right) dx$$

$$= \frac{1}{8} \left[6x - \frac{x^2}{2} - \frac{5}{2}x\right]_0^2$$

$$= \frac{1}{8} \left[12 - \frac{\cancel{4}^2}{\cancel{2}} - \frac{5}{\cancel{2}}.\cancel{2}\right] = \frac{1}{8}[5] = 5/8$$

$$\therefore \; p(X<1 / Y<3) = \frac{3/8}{5/8} = 3/5$$

(2) Given $f(x,y) = cx(x-y)$, $0<x<2$, $-x<y<x$ and 0 elsewhere

(a) Evaluate c

(b) Find $f(x)$ & $f(y)$

(c) Find $f(Y/x)$ and $f(X/y)$

(d) Are X & Y are independent?

Soln:

(a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\Rightarrow \int_0^2 \int_{-x}^{x} cx(x-y)\,dy\,dx = 1$$

$$\Rightarrow c\int_0^2 \int_{-x}^{x} (x^2 - xy)\,dy\,dx = 1$$

$$\Rightarrow c\int_0^2 \left[x^2 y - \frac{x y^2}{2}\right]_{-x}^{x} dx = 1$$

$$\Rightarrow c\int_0^2 \left\{x^3 - \frac{x^3}{2} - \left[-x^3 - \frac{x^3}{2}\right]\right\} dx = 1$$

$$\Rightarrow c\int_0^2 2x^3\, dx = 1$$

$$\Rightarrow c\left[\frac{\cancel{2}x^4}{\cancel{4}_2}\right]_0^2 = 1$$

$$\Rightarrow c[8] = 1$$

$$\Rightarrow c = 1/8$$

$$\therefore f(x,y) = \begin{cases} \frac{1}{8}[x^2 - xy], & 0<x<2,\ -x<y<x \\ 0, & \text{elsewhere} \end{cases}$$

b) To find $f(x)$:

$$f(x) = \int_{-x}^{x} \frac{1}{8}(x^2 - xy)\, dy$$

$$= \frac{1}{8}\left[x^2 y - x\frac{y^2}{2}\right]_{-x}^{x}$$

$$= \frac{1}{8}\left[x^3 - \frac{x^3}{2} - \left(-x^3 - \frac{x^3}{2}\right)\right]$$

$$= \frac{1}{8}[2x^3]$$

$$\therefore f(x) = \frac{x^3}{4}, \quad 0<x<2$$

To find $f(y)$:

$$f(y) = \int_0^2 \frac{1}{8}(x^2 - xy)\,dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3} - y\frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{8}\left[\frac{8}{3} - 2y\right]$$

$$= \frac{1}{3} - \frac{y}{4}$$

$$\therefore f(y) = \frac{4-3y}{12}, \quad -x < y < x$$

(c) To find $f(y/x)$:

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{8}(x^2 - xy)}{\frac{x^3}{4}} = \frac{1}{2x^2}(x - y)$$

To find $f(x/y)$:

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{\frac{1}{8}(x^2 - xy)}{\frac{1}{12}(4-3y)} = \frac{3}{2}\,\frac{x^2 - xy}{4-3y}$$

cd) Are x & y are independent :

WKT $f(x,y) = f(x) \cdot f(y)$

Here $\frac{1}{8}(x^2 - xy) \neq \frac{x^3}{4} \cdot \frac{1}{12}(4-3y)$

$\therefore$ x & y are not independent.