EXPONENTIAL DISTRIBUTION : [CONTINUOUS]

A continuous random variable x is said to follow exponential distribution if its probability density function is given by,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find Moment generating function, its Mean and its Variance using Exponential distribution.

MOMENT GENERATING FUNCTION:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x}\,dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty}$$

$$= \lambda \left\{ \frac{e^{-(\lambda-t)\infty} - e^{-(\lambda-t)0}}{-(\lambda-t)} \right\}$$

$$= \frac{\lambda}{\lambda - t}$$

$$\therefore M_x(t) = \frac{\lambda}{\lambda - t}$$

MEAN:

$$E(x) = \left\{ \frac{d}{dt} \left[M_x(t) \right] \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} \frac{\lambda}{\lambda - t} \right\}_{t=6}$$

$$= \left\{ -\lambda (\lambda - t)^{-2} (-1) \right\}_{t=0}$$

$$= \lambda (\lambda)^{-2}$$

$$= \lambda / \lambda^2 = 1/\lambda$$

$$\therefore E(x) = 1/\lambda$$

Now $E(x^2) = \left\{\frac{d^2}{dt^2} M_x(t)\right\}_{t=0}$

$$= \left\{\frac{d^2}{dt^2} \frac{\lambda}{\lambda-t}\right\}_{t=0}$$

$$= \left\{\frac{d}{dt} \left[\lambda(\lambda-t)^{-2}\right]\right\}_{t=0}$$

$$= \left\{\lambda(-2)(\lambda-t)^{-3}(-1)\right\}_{t=0}$$

$$= \left\{2\lambda(\lambda-t)^{-3}\right\}_{t=0}$$

$$= 2\lambda/\lambda^3 = 2/\lambda^2$$

VARIANCE:

$$Var(x) = E(x^2) - (E(x))^2$$

$$= 2/\lambda^2 - (1/\lambda)^2$$

$$= 1/\lambda^2$$

$\therefore Var(x) = 1/\lambda^2.$

MEMORYLESS PROPERTY:

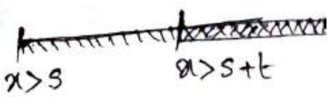
If x is an exponential distribution then

$$p(x>s+t/x>s) = p(x>t), \quad s,t \geq 0$$

Soln:

$$p(x>s+t/x>s) = \frac{p(x>s+t \cap x>s)}{p(x>s)}$$

$$= \frac{p(x>s+t)}{p(x>s)}$$



$$\text{Now } p(x>s) = \int_s^\infty \lambda e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_s^\infty$$

$$= \lambda \left[\frac{e^{-\lambda\infty} - e^{-\lambda s}}{-\lambda}\right]$$

$$= e^{-\lambda s}$$

$$\therefore p(x>s) = e^{-\lambda s}$$

$$\Rightarrow p(x>s+t) = e^{-\lambda(s+t)}$$

$$\therefore p(x > s+t / x > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= \frac{e^{-\lambda s} \cdot e^{-\lambda t}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$= p(x > t)$$

$\therefore p(X > s+t / x > s) = p(x > t)$. Hence proved

PROBLEMS:

1) If x is a random variable which follows an exponential distribution with parameter λ with $p(x \leq 1) = p(x > 1)$, find variance.

Soln:

Given: $p(x \leq 1) = p(x > 1)$

$$\Rightarrow \int_0^1 \lambda e^{-\lambda x} dx = \int_1^\infty \lambda e^{-\lambda x} dx$$

$$\Rightarrow \cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_0^1 = \cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_1^\infty$$

$$\Rightarrow \quad e^{-\lambda} - 1 = 0 - e^{-\lambda}$$

$$\Rightarrow \quad 2e^{-\lambda} = 1$$

$$\Rightarrow \quad e^{-\lambda} = 1/2$$

$$\Rightarrow \quad -\lambda = \log 1/2$$

$$\Rightarrow \quad -\lambda = \cancel{\log 1} - \log 2$$

$$\Rightarrow \quad \lambda = \log 2$$

$$\text{Now } var(x) = 1/\lambda^2$$

$$= 1/(\log 2)^2$$

$$= \frac{1}{2\log 2}$$

2) A mileage which car owner get with certain kind of radial tyre is a RV having exponential distribution with mean 4000 km. Find probability that one of these tyres will last

(i) atleast 2000 km
(ii) between 1000 to 4000 km
(iii) atmost 2000 km.

Soln:

Given: Mean $= 1/\lambda = 4000$

(ie) $\lambda = 1/4000$

WKT $f(x) = \lambda e^{-\lambda x}, \ x \geq 0$

$\therefore f(x) = \frac{1}{4000} e^{-\frac{x}{4000}}, \ x \geq 0$

(i) $p(\text{atleast } 2000 \text{ km}) = p(x \geq 2000)$

$$= \int_{2000}^{\infty} \frac{1}{4000} e^{-\frac{1}{4000}x} dx$$

$$= \frac{1}{\cancel{4000}} \left[\frac{e^{-\frac{1}{4000}x}}{-\frac{1}{\cancel{4000}}} \right]_{2000}^{\infty}$$

$$= -\frac{e^{-\frac{2000}{4000}}}{-1} = e^{-1/2}$$

(ii) $p(\text{between } 1000 \text{ to } 4000 \text{ km}) = p(1000 \leq x \leq 4000)$

$$= \int_{1000}^{4000} \frac{1}{4000} e^{-\frac{1}{4000}x} dx$$

$$= \frac{1}{\cancel{4000}} \left[\frac{e^{-\frac{1}{4000}x}}{-\frac{1}{\cancel{4000}}} \right]_{1000}^{4000}$$

$$= \frac{e^{-1} - e^{-1/4}}{-1}$$

$$= e^{-1/4} - e^{-1}$$

(iii) p (atmost 2000 km) $= p(x \le 2000)$

$$= \int_0^{2000} \frac{1}{4000} e^{-\frac{1}{4000}x} dx$$

$$= \frac{1}{\cancel{4000}} \left[\frac{e^{-\frac{1}{4000}x}}{-\frac{1}{\cancel{4000}}} \right]_0^{2000}$$

$$= \frac{e^{-1/2} - 1}{-1}$$

$$= 1 - e^{-1/2}$$

3) If x has an exponential distribution with mean 2 find $p(x<1/x<2)$

Soln:

Given : Mean $= 1/\lambda = 2$

(or) $\lambda = 1/2$

WKT $f(x) = \lambda e^{-\lambda x}, x \ge 0$

$\therefore f(x) = 1/2 \, e^{-1/2 x}, x \ge 0$

Now $p(x<1/x<2) = \frac{p(x<1 \cap x<2)}{p(x<2)}$

$= \frac{p(x<1)}{p(x<2)}$

Now $p(x<1) = \int_0^1 \frac{1}{2} e^{-\frac{1}{2}x} dx = \frac{1}{2}\left[\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_0^1$

$= \frac{e^{-\frac{1}{2}} - 1}{-1} = 1 - e^{-\frac{1}{2}}$

Now $p(x<2) = \int_0^2 \frac{1}{2} e^{-\frac{1}{2}x} dx = \frac{1}{2}\left[\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_0^2$

$= \frac{e^{-\frac{1}{2}\cdot 2} - 1}{-1} = 1 - e^{-1}$

$\therefore p(x<1/x<2) = \frac{p(x<1)}{p(x<2)}$

$= \frac{1-e^{-\frac{1}{2}}}{1-e^{-1}}$