UNIFORM DISTRIBUTION (OR) RECTANGULAR DISTRIBUTION

[CONTINUOUS]

Defn:

A random variable x is said to have a continuous uniform distribution over an interval (a,b) if its probability density function is a constant r.v., given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

where a and b are said to be two parameters of the uniform distribution on (a,b)

Note:

The pdf of a uniform variable 'x' in $(-a,a)$ is given by

$$f(x) = \begin{cases} \frac{1}{2a}, & -a<x<a \\ 0, & \text{otherwise} \end{cases}$$

Find Moment generating Function, Mean and its variance using uniform distribution.

Moment generating function:

The moment generating function of uniform distribution is

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx} \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{b-a} \left[\frac{e^{tb} - e^{ta}}{t}\right]$$

$$\therefore M_x(t) = \frac{1}{b-a} \cdot \frac{e^{tb} - e^{ta}}{t}$$

Mean :-

$$E(x) = \left\{\frac{d}{dt} M_x(t)\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left[\frac{e^{tb} - e^{ta}}{(b-a)t}\right]\right\}_{t=0}$$

$$= \frac{1}{b-a}\left\{\frac{t(e^{tb}.b - e^{ta}.a) - (e^{tb} - e^{ta})}{t^2}\right\}_{t=0}$$

$$= \frac{0}{0} \quad \text{[Apply L-Hospital's rule]}$$

$$= \frac{1}{b-a}\left\{\frac{t(e^{tb}.b^2 - e^{ta}.a^2) + (e^{tb}.\cancel{b} - e^{ta}.a) - (e^{tb}.\cancel{b} - e^{ta}.a)}{2t}\right\}_{t=0}$$

$$= \frac{0}{0} \quad \text{[Apply L'Hospital's rule]}$$

$$= \frac{1}{b-a}\left\{\frac{t(e^{tb}.b^3 - e^{ta}.a^3) + (e^{tb}.b^2 - e^{ta}.a^2)}{2}\right\}_{t=0}$$

$$= \frac{1}{b-a}\left\{\frac{b^2 - a^2}{2}\right\} = \frac{a+b}{2}$$

$$\therefore E(x) = \frac{a+b}{2}$$

Another Method:

WKT $f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$

Mean:

$$E(x) = \int_a^b x \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\int_a^b x\, dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{b+a}{2}$$

Now $E(x^2) = \int_a^b x^2 \frac{1}{b-a}\, dx$

$$= \frac{1}{b-a}\int_a^b x^2\, dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a} \cdot \frac{b^3-a^3}{3}$$

$$= \frac{1}{\cancel{b-a}} \cdot \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3}$$

$$= \frac{b^2+ab+a^2}{3}$$

Variance:

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{(b-a)^2}{12}$$

PROBLEMS:

(1) If x is uniformly distributed with mean 1 and variance $4/3$. Find $p(x<0)$.

Soln:

WKT $f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{elsewhere} \end{cases}$

Given: Mean $=1$; variance $=4/3$

$\Rightarrow$ Mean $=1$; $\frac{(b-a)^2}{12} = 4/3$

$\Rightarrow \frac{b+a}{2} = 1$; $(b-a)^2 = 16$

$\Rightarrow b+a = 2$ —(1); $b-a = 4$ —(2)

Solving (1) & (2) we get

$b=3$; $a=-1$

$$\therefore f(x) = \begin{cases} 1/4 & , -1<x<3 \\ 0 & , \text{otherwise} \end{cases}$$

Now $p(x<0) = \int_{-1}^{0} 1/4 \, dx$

$= 1/4 [x]_{-1}^{0} = 1/4$

(2) Show that for uniform distribution $f(x) = \frac{1}{2a}$, $-a<x<a$, the momentum about the origin is $\frac{\sin hat}{at}$.

Soln:

WKT $f(x) = \begin{cases} 1/2a & , -a<x<a \\ 0 & , \text{otherwise} \end{cases}$ (given)

Now $M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx$

$= \int_{-a}^{a} e^{tx} \frac{1}{2a} . dx$

$$= \frac{1}{2a}\left[\frac{e^{tx}}{t}\right]_{-a}^{a}$$

$$= \frac{1}{2a}\left[\frac{e^{at}-e^{-at}}{t}\right]$$

$$= \frac{1}{\cancel{2}a}\;\frac{\cancel{2}\sinh at}{t}$$

$$= \frac{\sinh at}{at}, \text{ hence proved.}$$

3) A random variable x has an uniform distribution over interval (-3, 3). Compute $p(x=2)$, $p(x<2)$, $p(|x|<2)$, $p(|x-2|<2)$. Find k such that $p(x>k) = 1/3$.

Soln:

WKT $f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$

Given: the interval $(-3, 3)$ (ie) $a=-3$; $b=3$

$$\Rightarrow f(x) = \begin{cases} 1/6, & -3<x<3 \\ 0, & \text{otherwise} \end{cases}$$

(i) $p(x=2) = 0$

(ii) $p(x<2) = \int_{-3}^{2} \frac{1}{6}\, dx$

$= \frac{1}{6}[x]_{-3}^{2} = \frac{1}{6}[2+3]$

$= 5/6$

(iii) $p(|x|<2) = p(-2<x<2)$

$= \int_{-2}^{2} \frac{1}{6}\, dx$

$= \frac{1}{6}[2+2] = 4/6$.

(iv) $p(|x-2|<2) = p(-2 < x-2 < 2)$

$= p(0<x<4)$

$= \int_{0}^{3} \frac{1}{6}\, dx + \int_{3}^{4} 0\, dx$

$= \frac{1}{6}[3] = \frac{1}{2}$

(v) $p(x>k) = \frac{1}{3}$

$\Rightarrow \int_{k}^{3} \frac{1}{6}\, dx = \frac{1}{3}$

$\Rightarrow \frac{1}{6}[x]_k^3 = \frac{1}{3}$

$\Rightarrow \frac{1}{6}[3-k] = \frac{1}{3}$

$\Rightarrow 3-k = 2$

$\Rightarrow k = 1$

4) A no. of pc sold daily from computer world is uniformly distributed with minimum 2000 pc & maximum 5000 pc. Find

(i) probability that daily sales will fall between 2500 and 3000 pc.

(ii) what is the probabability that computer world will sell atleast 4000 pc.

(iii) What is the probability that computer world will sell 2500 pc.

Soln:

WKT $f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise.} \end{cases}$

Given: Minimum - 2000 & Maximum - 5000

(ii) $a = 2000$; $b = 5000$

$$\Rightarrow f(x) = \begin{cases} 1/3000 , & 2000 < x < 5000 \\ 0 , & \text{otherwise} \end{cases}$$

(i) $$p(2500 < x < 3000) = \int_{2500}^{3000} 1/3000 \, dx$$

$$= 1/3000 \, [x]_{2500}^{3000}$$

$$= 1/3000 \, [3000 - 2500]$$

$$= 1/6$$

(ii) $$p(x \geq 4000) = \int_{4000}^{5000} 1/3000 \, dx$$

$$= 1/3000 \, [x]_{4000}^{5000}$$

$$= 1/3000 \, [5000 - 4000] = 1/3$$

(iii) $p(x = 2500) = 0$