CONTINUOUS RANDOM VARIABLE

The density function of a random variable x is given by $f(x) = kx(2-x)$, $0 \leq x < 2$. Find k, mean, variance and 3^{rd} moment.

Soln:

(i) To find k:

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{2} kx(2-x)\,dx = 1$$

$$\Rightarrow k\int_{0}^{2} (2x - x^2)\,dx = 1$$

$$\Rightarrow k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow k\left[4 - \frac{8}{3}\right] = 1$$

$$\Rightarrow k = 3/4$$

$$\therefore f(x) = \frac{3}{4}x(2-x)$$

(ii) To find mean:

WKT $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 x \cdot \frac{3}{4} x(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$

$$= \frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = 1$$

(iii) To find Variance:

Now $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^2 x^2 \cdot \frac{3}{4} x(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx$

$$= \frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{6}{5}$$

$\therefore Var(X) = E(X^2) - (E(X))^2 = \frac{6}{5} - 1 = 1/5$

(iv) To find 3rd moment:

$E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)\,dx = \int_0^2 x^3 \cdot \frac{3}{4} x(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^4 - x^5)\,dx$

$$= \frac{3}{4}\left[2\frac{x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{8}{5}$$

The p.d.f of a cts. R.V x is $f(x) = ke^{-|x|}$. Find k & $F(x)$

Soln:

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} ke^{-|x|}\,dx = 1$$

$$2k\int_{0}^{\infty} e^{-x}\,dx = 1$$

$$2k \left.\frac{e^{-x}}{-1}\right]_{0}^{\infty} = 1$$

$$2k = 1 \Rightarrow k = 1/2$$

$$\therefore f(x) = \frac{1}{2}e^{-|x|}$$

Now $f(x) = \frac{1}{2}e^{-|x|} = \begin{cases} \frac{1}{2}e^{-(-x)}, & -\infty < x < 0 \\ \frac{1}{2}e^{-(x)}, & 0 < x < \infty \end{cases}$

$$= \begin{cases} \frac{1}{2}e^{x}, & -\infty < x < 0 \\ \frac{1}{2}e^{-x}, & 0 < x < \infty \end{cases}$$

(ii) For $x \le 0$,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{1}{2}e^{x}\,dx = \frac{1}{2}e^{x}\Big]_{-\infty}^{x} = \frac{e^{x}}{2}$$

For $x > 0$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} \tfrac{1}{2} e^{x}\,dx + \int_{0}^{x} \tfrac{1}{2} e^{-x}\,dx = \frac{1}{2} e^{x}\Big]_{-\infty}^{0} + \frac{1}{2} \frac{e^{-x}}{-1}\Big]_{0}^{x}$$

$$= \frac{1}{2} + \frac{1}{2}\left[e^{-x} + 1\right]$$

$$= \frac{1}{2}\left[2 - e^{-x}\right]$$

A continuous random variable x has the distribution function $F(x) = \begin{cases} 0 & , x \le 1 \\ k(x-1)^4, & 1 < x \le 3 \\ 0 & , x > 3 \end{cases}$. Find pdf: $f(x)$, k. and $P[X < 2]$.

Soln:

(i) If $F(x)$ given then to find $f(x)$, WKT

$$f(x) = \frac{d}{dx}\left[F(x)\right]$$

$$= \frac{d}{dx}\left[k(x-1)^4\right] = 4k(x-1)^3$$

$$\therefore f(x) = \begin{cases} 0, & x \le 1 \\ 4k(x-1)^3, & 1 < x \le 3 \\ 0, & x > 3 \end{cases}$$

(ii) To find k:

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\int_{1}^{3} 4k(x-1)^3\,dx = 1$$

$$4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$\Rightarrow k(2^4) - k(0) = 1$$

$$\Rightarrow 16k = 1 \Rightarrow k = 1/16$$

$$\therefore f(x) = \begin{cases} 0 & , x \le 1 \\ \frac{1}{4}(x-1)^3, & 1 < x \le 3 \\ 0 & , x > 3 \end{cases}$$

(iii) $P[x<2]$:

WKT $P[x \le x] = F[x]$

$P[x<2] = F[2]$

$= K[2-1]^4 = \frac{1}{16}$

5) Find MGF of an exponential random variable & hence find the mean & variance.

Soln: For exponential random variable, WKT

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

To find MGF:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^\infty$$

$$= \frac{\lambda}{\lambda - t}$$

To find Mean:

$$E(x) = \mu_1' = \left\{ \frac{d}{dt} [M_x(t)] \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} \left(\frac{\lambda}{\lambda - t} \right) \right\}_{t=0}$$

$$= \left\{ \frac{\lambda}{(\lambda-t)^2} \right\}_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$\therefore E(x) = \frac{1}{\lambda}$$

To find Variance :

Now , $E(x^2) = \mu_2' = \left\{ \frac{d^2}{dt^2} M_X(t) \right\}_{t=0}$

$$= \left\{ \frac{d}{dt} \frac{\lambda}{(\lambda - t)^2} \right\}_{t=0}$$

$$= \left\{ \frac{2\lambda}{(\lambda - t)^3} \right\}_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$\therefore \ Var(x) = E(x^2) - (E(x))^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

MATHEMATICAL EXPECTATIONS :

Let 'x' be a random variable with probability density function $f(x)$, or probability mass function $p(x)$ then the mathematical expectation of 'x' is denoted by $E(x)$ and is given by

$$E(x) = \sum_x x\,p(x), \text{ for a discrete random Variable}$$

$$= \int_{-\infty}^{\infty} x f(x)\,dx, \text{ for a Continuous random Var}$$

The variance of a random variable 'x' is denoted by $\text{var}(x)$ and is defined by

$$\text{var}(x) = E(x^2) - (E(x))^2$$

$$= \sum_x x^2 p(x) - \left[\sum_x x\,p(x)\right]^2, \text{ for a discrete RV.}$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2, \text{ for a cts. RV.}$$

PROPERTIES :

If 'x' and 'y' are random variable and a, b are constants, then

(i) $E(a) = a$

(ii) $E(ax) = a\,E(x)$

(iii) $E(ax+b) = a\,E(x)+b$

(iv) If $y \le x$, then $E(y) \le E(x)$

(v) If x & y are independent then $E(xy) = E(x) \cdot E(y)$

(vi) $E(x^2) \geq (E(x))^2$

(vii) $Var(x) \geq 0$

(viii) $Var(a) = 0$

(ix) $Var(ax) = a^2 var(x)$

(x) $Var(x \pm a) = var(x)$

(xi) $Var(ax+b) = a^2 Var(x)$

(xii) $Var(ax+by) = a^2 var(x) + b^2 var(y)$

PROBLEMS:

(1) When a die is thrown, 'x' denotes the number that turns up. Find $E(x)$, $E(x^2)$ and $var(x)$.

Soln:

Let 'x' denotes the number that turns up in die.

(i.e.) 'x' takes values 1, 2, 3, 4, 5, 6.

x:	1	2	3	4	5	6
$p(x)$:	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

Now $E(x) = \sum_x x\,p(x)$

$= 1\times 1/6 + 2\times 1/6 + 3\times 1/6 + 4\times 1/6 + 5\times 1/6 + 6\times 1/6$

$= 21/6 = 7/2$

$E(x^2) = \sum_x x^2 p(x)$

$= (1)^2\times 1/6 + (2)^2\times 1/6 + (3)^2\times 1/6 + (4)^2\times 1/6 + (5)^2\times 1/6 + (6)^2\times 1/6$

$= 91/6$

$Var(x) = E(x^2) - (E(x))^2$

$= 91/6 - (21/6)^2$

$= \dfrac{546-441}{36} = \dfrac{105}{36}$

$= \dfrac{35}{12}$

Find $E(x^2)$ and $E(2x+3)$ for the following probability distribution:

x:	-2	-1	0	1	2	3
$p(x)$:	$1/6$	$1/6$	0	$2/3$	$1/6$	$1/6$

Soln:

Now $E(x^2) = \sum_x x^2 p(x)$

$$= (-2)^2 \times 1/6 + (-1)^2 \times 1/6 + 0 + (1)^2 \times 2/6 + (2)^2 \times 1/6 + (3)^2 \times 1/6$$

$$= \frac{4+1+4+4+9}{6} = \frac{22}{6} = \frac{11}{3}$$

$E(2x+3) = 2E(x) + 3$

Now $E(x) = \sum_x x\,p(x)$

$$= (-2) \times 1/6 + (-1) \times 1/6 + 0 + 1 \times 2/3 + 2 \times 1/6 + 3 \times 1/6$$

$$= \frac{-2-1+4+2+3}{6} = 1$$

$\therefore E(2x+3) = 2(1) + 3$

$= 5$

Let x be a random variable with $E(x)=1$ and $E(x(x-1))=4$. Find $var(x)$ and $var(2-3x)$

Soln:

$E(x)=1$; $E(x(x-1))=4$ (given)

Now $E(x(x-1))=4$

$E(x^2-x)=4$

$E(x^2)-E(x)=4$

$E(x^2)-1=4$

$\Rightarrow E(x^2)=5$

$\therefore var(x)=E(x^2)-(E(x))^2=5-1=4$

Now $Var(2-3x)=(-3)^2 var(x)=9\times(4)=36$.