DISCRETE RANDOM VARIABLE

Given the following probability distribution, compute $p(|x|\le 1)$, $p(|x|>2)$, $p(2x+3\le 5)$

x	-3	-2	-1	0	1	2	3
$p(x)$	$1/6$	$1/6$	$1/6$	$1/12$	$1/12$	$1/6$	$1/6$

Soln:

$$p(|x|\le 1) = p(-1\le x\le 1)$$
$$= p(x=-1)+p(x=0)+p(x=1)$$
$$= 1/6 + 1/12 + 1/12 = 4/12 = 1/3$$

$$p(|x|>2) = 1-p(|x|\le 2)$$
$$= 1-p(-2\le x\le 2)$$
$$= 1-[p(x=-2)+p(x=-1)+p(x=0)+p(x=1)+p(x=2)]$$
$$= 1-[1/6+1/6+1/12+1/12+1/6]$$
$$= 4/12 = 1/3$$

$$p(2x+3\le 5) = p(2x\le 5-3) = p(x\le 1)$$
$$= 1-p(x>1) = 1-[p(x=2)+p(x=3)]$$
$$= 1-[1/6+1/6] = 2/3$$

RESULTS :

(i) $p(a < x \le b) = F(b) - F(a)$

(ii) $p(a \le x \le b) = F(b) - F(a) + p(x=a)$

(iii) $p(a < x < b) = F(b) - F(a) - p(x=b)$

(iv) $p(a \le x < b) = F(b) - F(a) - p(x=b) + p(x=a)$

Let x be a discrete random variable whose Cumulative distribution is

$$F(x) = \begin{cases} 0, & x < -3 \\ 1/6, & -3 \le x < 6 \\ 1/2, & 6 \le x < 10 \\ 1, & x \ge 10 \end{cases}$$

(i) Find the probability distribution of x

(ii) Find $p(x \le 4)$, $p(-5 < x \le 4)$, $p(2 < x < 7)$, $p(x > 4)$

(iii) Find Mean & Variance of x.

Soln:

(i) Probability distribution of x :

WKT, if $F(x)$ is given then to find pmf.

$$p(x=x_i) = F[x_i] - F[x_{i-1}]$$

x :	-4	-3	6	10
$F(x)$:	0	$1/6$	$1/2$	1
$p(x)$:	0	$1/6$	$1/3$	$1/2$

ii) (a) To find $p(x \le 4)$:

WKT $p(x \le x) = F(x)$

$\Rightarrow p(x \le 4) = F(4) = 1/6$

(b) $p(-5 < x \le 4) = F(4) - F(-5)$

$= 1/6 - 0 = 1/6$

(c) $p(2 < x < 7) = F(7) - F(2) - p(x=7)$

$= 1/2 - 1/6 - 0 = 1/3$

(or) $p(2 < x < 7) = p(x=6) = 1/3$

(d) $p(x>4) = 1-p(x\le 4) = 1-\frac{1}{6} = \frac{5}{6}$

(iii) Mean & variance:

Mean: $E(x) = \sum_i x_i\, p(x_i)$

$= (-4)(0) + (-3)(\frac{1}{6}) + (6)\times(\frac{1}{3}) + 10\times\frac{1}{2}$

$= -\frac{1}{2} + 2 + 5 = \frac{13}{2}$

Now $E(x^2) = \sum_i x_i^2\, p(x_i)$

$= (-4)^2(0) + (-3)^2(\frac{1}{6}) + (6)^2(\frac{1}{3}) + (10)^2(\frac{1}{2})$

$= 9\times\frac{1}{6} + 36\times\frac{1}{3} + 100\times\frac{1}{2}$

$= \frac{127}{2}$

$Var(x) = E(x^2) - (E(x))^2 = \frac{127}{2} - \left(\frac{13}{2}\right)^2$

$= \frac{85}{4}$

) If the cumulative distribution function of a R.V. X is given by $F(x) = \begin{cases} 1 - 4/x^2, & x > 2 \\ 0, & x \le 2 \end{cases}$

Find (i) $p(X < 3)$ (ii) $p(4 < X < 5)$ (iii) $p(X \ge 3)$

Soln:

(i) $p(X < 3) = p(X \le 3) - p(X = 3)$

$= F(3) - p(X = 3)$

$= \left(1 - \frac{4}{3^2}\right) - 0$

$= 1 - 4/9 = 5/9$

(ii) $p(4 < X < 5) = F(5) - F(4) - p(X = 5)$

$= \left(1 - \frac{4}{5^2}\right) - \left(1 - \frac{4}{4^2}\right) - 0$

$= \left(1 - 4/25\right) - \left(1 - 4/16\right)$

$= 9/100$

(iii) $p(X \ge 3) = 1 - p(X < 3) = 1 - 5/9 = 4/9$

) If a random variable takes the values 1, 2, 3, 4 such that $2p(x=1) = 3p(x=2) = p(x=3) = 5p(x=4)$. Find the probability distribution of x.

Soln:

Given: $x = 1, 2, 3, 4$

Let $2p(x=1) = 3p(x=2) = p(x=3) = 5p(x=4) = k$

(or) $p(x=1) = k/2$

$p(x=2) = k/3$

$p(x=3) = k$

$p(x=4) = k/5$

WKT $\sum_i p(x_i) = 1$

$p(x=1) + p(x=2) + p(x=3) + p(x=4) = 1$

$k/2 + k/3 + k + k/5 = 1$

$\frac{15k + 10k + 30k + 6k}{30} = 1 \Rightarrow \frac{61k}{30} = 1$

$\Rightarrow k = \frac{30}{61}$

$\therefore\ p(x=1) = \frac{30}{61} \times \frac{1}{2} = 15/61$

$p(x=2) = \frac{30}{61} \times \frac{1}{3} = 10/61$

$p(x=3) = \frac{30}{61}$

$p(x=4) = \frac{30}{61} \times \frac{1}{5} = 6/61$

$\therefore$ probability distribution of x:

$X = x$:	1	2	3	4
$p(x)$:	15/61	10/61	30/61	6/61

A R.V. x takes the values $-3, -2, -1, 0, 1, 2, 3$ such that $p(x=0) = p(x>0) = p(x<0)$. $p(x=-3) = p(x=-2) = p(x=-1) = p(x=1) = p(x=2) = p(x=3)$. Obtain the probability distri. and distri. function of x.

Soln:

Let $p(x=0) = p(x>0) = p(x<0) = k$

(ie) $p(x=0) = k$; $p(x>0) = k$; $p(x<0) = k$

WKT $\sum_i p(x_i) = 1$

$k + k + k = 1$

$\Rightarrow k = 1/3$

$\therefore p(x=0) = 1/3$; $p(x>0) = 1/3$; $p(x<0) = 1/3$

Let $p(x=1) = p(x=2) = p(x=3) = k_1$

from the above $p(x>0) = 1/3$

$\Rightarrow p(x=1) + p(x=2) + p(x=3) = 1/3$

$\Rightarrow k_1 + k_1 + k_1 = 1/3 \Rightarrow k_1 = 1/9$

$\therefore p(x=1) = p(x=2) = p(x=3) = 1/9$

Let $p(x=-3) = p(x=-2) = p(x=-1) = k_2$

from the above WKT $p(x<0) = 1/3$

$p(x=-3) + p(x=-2) + p(x=-1) = 1/3$

$k_2 + k_2 + k_2 = 1/3 \Rightarrow k_2 = 1/9$

$\therefore p(x=-3) = p(x=-2) = p(x=-1) = 1/9$

$\therefore$ probab. distri of x :

x :	-3	-2	-1	0	1	2	3
$p(x)$:	$1/9$	$1/9$	$1/9$	$1/3$	$1/9$	$1/9$	$1/9$

Cumulative distribution of x :

x :	-3	-2	-1	0	1	2	3
$F(x)$:	$1/9$	$2/9$	$3/9$	$6/9$	$7/9$	$8/9$	$9/9 = 1$

Find the MGF of the RV x whose probability func. $p(x=x) = \frac{1}{2^x}$, $x = \cancel{0}\, 1, 2, \ldots$ Find its mean & Variance.

Soln:

$$M_x(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$= \frac{e^t}{2}\left[1+\frac{e^t}{2}+\left(\frac{e^t}{2}\right)^2+\cdots\right] = \frac{e^t}{2}\left[1-\frac{e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\cdot\frac{2}{2-e^t} = \frac{e^t}{2-e^t}$$

To find mean & variance :

$$\underset{(\mu_1')}{\text{Mean}} = \left\{\frac{d}{dt}\left[M_x(t)\right]\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left[\frac{e^t}{2-e^t}\right]\right\}_{t=0} = \left\{\frac{d}{dt}\left[e^t(2-e^t)^{-1}\right]\right\}_{t=0}$$

$$= \left[e^t(-1)(2-e^t)^{-2}(-e^t)+(2-e^t)^{-1}e^t\right]_{t=0}$$

$$= (-1)(2-1)^{-2}(-1)+(2-1)^{-1}(1)$$

$$= (1)^{-2}+(1)^{-1} = 2$$

Now $\mu_2' = \left\{\frac{d^2}{dt^2}\left[M_x(t)\right]\right\}_{t=0}$

$$= \left\{\frac{d^2}{dt^2}\left[\frac{e^t}{2-e^t}\right]\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left[e^{2t}(2-e^t)^{-2} + (2-e^t)^{-1}e^t\right]\right\}_{t=0}$$

$$= \Big\{\left[e^{2t}(-2)(2-e^t)^{-3}(-e^t) + (2-e^t)^{-2}e^{2t}\cdot 2\right] + \left[(2-e^t)^{-1}e^t + e^t(-1)(2-e^t)^{-2}(e^t)\right]\Big\}_{t=0}$$

$$= \Big\{2e^{3t}(2-e^t)^{-3} + 2e^{2t}(2-e^t)^{-2} + (2-e^t)^{-1}e^t + e^{2t}(2-e^t)^{-2}\Big\}_{t=0}$$

$$= 2(2-1)^{-3} + 2(2-1)^{-2} + (2-1)^{-1} + (2-1)^{-2}$$

$$= 2(1)^{-3} + 2(1)^{-2} + (1)^{-1} + (1)^{-2}$$

$$= 4+2$$

$\therefore$ Variance $= \mu_2' - \mu_1^2 = 6 - (2)^2 = 2$

Let x be the RV with probability law $p(x=r) = q^{r-1}p$; $r = 1, 2, 3 \ldots$ Find MGF and also mean; variance assuming $p+q=1$