CONTINUOUS RANDOM VARIABLE

Continuous

(1) DEFINITION:

A random variable x is said to be continuous if it takes all possible values in an interval.

(2) PROBABILITY DENSITY FUNCTION [PDF]

If x is a continuous random variable then the function $f(x)$ is called p.d.f of x provided $f(x)$ satisfies the following conditions:

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Moreover, $p(a \leq x \leq b)$ (or) $p(a < x < b)$ is defined as

$$p(a \leq x \leq b) = \int_a^b f(x)\,dx.$$

$$p(x < a) \text{ (or) } p(x \leq a) = \int^{a} f(x)\,dx$$

(lower limit from prob.)

$$p(x > a) \text{ (or) } p(x \geq a) = \int_a f(x)\,dx.$$

(upper limit) from prob.)

(3) TO FIND CONSTANTS [k, a, c ...]

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

(4) CUMULATIVE DISTRIBUTION FUNCTION (or) DISTRIBUTION FUNCTION

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

If cumulative distribution is given then to find p.d.f, $f(x) = \frac{d}{dx}[F(x)]$

(5) TO FIND MEAN (or) FIRST MOMENT:

$$E(X) = \mu_1' = \int_{-\infty}^{\infty} x f(x)\,dx$$

6) TO FIND SECOND MOMENT:

$$E(X^2) = \mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

7) TO FIND VARIANCE :

$$Var(x) = E(x^2) - (E(x))^2$$
$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2$$

8) TO FIND MOMENTS :

$\mu_1' = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$, first moment

$\mu_2' = E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$, second moment

$\mu_3' = E(x^3) =$, Third moment

$\mu_4' = E(x^4) =$, fourth moment

. . . . . .

$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$, r^{th} moment.

9) MOMENT GENERATING FUNCTION [MGF] :

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx.$$

If the PDF of a continuous random variable X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the value of a.

(ii) Find cumulative distribution.

(iii) Compute $P[X \le 1.5]$

Soln:

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx = 1$$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$\left[\frac{ax^2}{2}\right]_0^1 + \left[ax\right]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$\frac{a}{2} + \left[2a - a\right] + \left[9a - \frac{9a}{2} - \left(6a - \frac{4a}{2}\right)\right] = 1$$

$$\Rightarrow 2a = 1$$

$$\Rightarrow a = \frac{1}{2}$$

$$\therefore f(x) = \begin{cases} x/2, & 0 \le x \le 1 \\ 1/2, & 1 \le x \le 2 \\ 3/2 - x/2, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(ii) Cumulative distribution:

For $x < 0$,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = 0$$

For $0 \le x \le 1$,

$$F(x) = P(X \le x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} x/2\,dx = \left.\frac{x^2}{4}\right]_0^x = \frac{x^2}{4}$$

For $1 \le x \le 2$,

$$F(x) = P(X \le x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} x/2\,dx + \int_{1}^{x} 1/2\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^x$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2} \quad = \frac{x}{2} - \frac{1}{4}$$

For $2 \le x \le 3$,

$$F(x) = P(X \le x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^x f(x)\,dx$$

$$= 0 + \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^2 + \left[\frac{3x}{2} - \frac{x^2}{4}\right]_2^x$$

$$= \frac{1}{4} + 1 - \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - 3 + 1$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

For $x > 3$

$$F(x) = P(X \le x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^{\infty} f(x)\,dx$$

$$= 0 + \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx + 0$$

$$= \frac{1}{4} + 1 - \frac{1}{2} + \frac{9}{2} - \frac{9}{4} - 3 + 1 = 1$$

$$\therefore F(x) = \begin{cases} 0, & x < 0 \\ x^2/4, & 0 \le x \le 1 \\ x/2 - 1/4, & 1 \le x \le 2 \\ \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4}, & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

(iii) $P[X \le 1.5] = P(0 \le x \le 1) + P(1 \le x \le 1.5)$

$$= \int_0^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx$$

$$= \int_0^1 x/2\,dx + \int_1^{1.5} 1/2\,dx = \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^{1.5}$$

$$= \frac{1}{4} + \frac{1.5}{2} - \frac{1}{2} = \frac{1}{2}$$

2) A continuous random variable x has a PDF

$f(x) = 3x^2, \ 0 \le x \le 1$.

(i) Find 'a' such that $p(x \le a) = p(x > a)$

(ii) Find 'b' such that $p(x > b) = 0.05$

(iii) Find 'c' such that $p(x > c) = 0.1$

(iv) Find 'k' such that $p(x < k) = 0.05$

(v) Find $p(0.2 < x < 0.7)$

(vi) Find $p(x \le 1/2 \,/\, 1/3 < x < 2/3)$

(vii) Find distribution function of x.

Soln:

WKT $p(x \le x) = \int_{-\infty}^{x} f(x)\,dx$

(i) Consider $p(x \le a) = \int_{-\infty}^{a} f(x)\,dx = \int_{0}^{a} 3x^2\,dx = x^3\Big]_0^a = a^3$

Now $p(x > a) = 1 - p(x \le a) = 1 - a^3$

Given: $p(x \le a) = p(x > a)$

$a^3 = 1 - a^3$

$2a^3 = 1 \Rightarrow a^3 = 1/2$

$\Rightarrow a = (1/2)^{1/3} = 0.793$.

(ii) Consider $p(x>b) = 0.05$.

$$\int_b^1 f(x)\,dx = 0.05$$

$$\int_b^1 3x^2\,dx = 0.05$$

$$x^3\Big]_b^1 = 0.05$$

$$1 - b^3 = 0.05$$

$$b^3 = 1 - 0.05 = 0.95$$

$$b = (0.95)^{1/3} = 0.983$$

(iii) Consider $p(x>c) = 0.1$

$$\int_c^1 f(x)\,dx = 0.1$$

$$\int_c^1 3x^2\,dx = 0.1$$

$$x^3\Big]_c^1 = 0.1$$

$$1 - c^3 = 0.1$$

$$c^3 = 0.9 \Rightarrow c = (0.9)^{1/3} = 0.965$$

(iv) Consider $p(x<k) = 0.05$

$$\int_0^k f(x)\,dx = 0.05.$$

$$\int_0^k 3x^2\,dx = 0.05$$

$$\left. x^3 \right]_0^k = 0.05$$

$$k^3 = 0.05 \Rightarrow k = (0.05)^{1/3} = 0.368$$

(v) $p[0.2 < x < 0.7]$;

$$\int_{0.2}^{0.7} f(x)\,dx = \int_{0.2}^{0.7} 3x^2\,dx = \left. x^3 \right]_{0.2}^{0.7} = (0.7)^3 - (0.2)^3$$

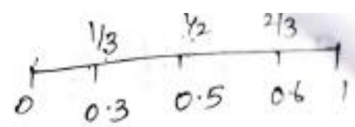
$$= 0.343 - 0.008$$

$$= 0.335$$

(vi) $p(x \le 1/2 \,/\, 1/3 < x < 2/3)$

$$= \frac{P[(x \le 1/2) \cap (1/3 < x < 2/3)]}{P(1/3 < x < 2/3)}$$

$$= \frac{P[(x \le 0.5) \cap (0.3 < x < 0.6)]}{p(0.3 < x < 0.6)}$$



$$= \frac{p(0.3 < x \le 0.5)}{p(0.3 < x < 0.6)}$$

Now $p(0.3 < x \leq 0.5) = \int_{0.3}^{0.5} f(x)\,dx = \int_{0.3}^{0.5} 3x^2\,dx = x^3\Big]_{0.3}^{0.5}$

$= (0.5)^3 - (0.3)^3 = 0.098$

Now $p(0.3 < x < 0.6) = \int_{0.3}^{0.6} f(x)\,dx = \int_{0.3}^{0.6} 3x^2\,dx = x^3\Big]_{0.3}^{0.6}$

$= (0.6)^3 - (0.3)^3 = 0.189$

$\therefore P\left[x \leq 1/2 \,/\, 1/3 < x < 2/3\right] = \frac{0.098}{0.189} = 0.5185$

(vii) Distribution function of x :

$F(x) = p(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$

For $x < 0$

$F(x) = p(X \leq x) = \int_{-\infty}^{x} f(x)\,dx = 0$

For $0 \leq x \leq 1$

$F(x) = p(X \leq x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx.$

$= 0 + \int_{0}^{x} 3x^2\,dx = x^3\Big]_{0}^{x} = x^3$

For $x > 1$.

$$F(x) = P(X \le x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{1} 3x^2\,dx + 0 = x^3\Big]_0^1 = 1$$

$$\therefore F(x) = \begin{cases} 0, & x < 0 \\ x^3, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$