Moment Generating function :

The moment generating function (m.g.f) of a random variable 'x' (about origin) whose probability function is given by,

$$M_X(t) = E[e^{tx}]$$

For a discrete random variable, m.g.f is given by

$$M_X(t) = \sum e^{tx} p(x)$$

For a continuous random variable, m.g.f is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

Problems :

(1) Prove that the r^{th} moment of the r.v 'X' about origin is $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$

Solution :

$$M_X(t) = E(e^{tx})$$

$$= E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \cdots \frac{t^r x^r}{r!} + \cdots\right]$$

$$= E(1) + E\left[\frac{tx}{1!}\right] + E\left[\frac{t^2x^2}{2!}\right] + \cdots + E\left[\frac{t^r x^r}{r!}\right] + \cdots$$

$$= 1 + t\,E[X] + \frac{t^2}{2!} E(X^2) + \cdots + \frac{t^r}{r!} E(X^r) + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$$\boxed{M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'}$$

(2) A random variable X has the probability function

$$f(x) = \frac{1}{2^x}, \quad x = 1, 2, 3, \ldots \infty$$

Find its (i) M.G.F (ii) Mean & Variance (iii) $P(x \text{ is even})$ (iv) $P(X \geq 5)$ and $P(X \text{ is divisible by } 3)$

Solution:

(i) $\text{M.G.F} = M_X(t) = E(e^{tx})$

$$= \sum_{x=1}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{2^x}$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right]$$

$$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1}$$

$$= \frac{e^t}{\cancel{2}} \cdot \frac{\cancel{2}}{2 - e^t}$$

$$\boxed{M_X(t) = \frac{e^t}{2 - e^t}}$$

(ii) $\text{Mean} = E(X) = M_X'(0)$

$$M_X'(t) = \frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2}$$

$$= \frac{2e^t - e^{2t} + e^{2t}}{(2 - e^t)^2}$$

$$M_X'(t) = \frac{2e^t}{(2 - e^t)^2}$$

$$E(X) = M_X'(0) = \frac{2}{1} = 2$$

$$\boxed{E(X) = 2}$$

$$M_X''(t) = \frac{(2-e^t)^2(2e^t) - 2e^t \cdot 2(2-e^t)(-e^t)}{(2-e^t)^4}$$

$$= \frac{2e^t(2-e^t)\left[2-e^t+2e^t\right]}{(2-e^t)^4}$$

$$= \frac{2e^t(2+e^t)}{(2-e^t)^3}$$

$$M_X''(0) = \frac{2(2+1)}{(2-1)^3} = 2(3) = 6$$

& $\underline{E(X^2) = 6}$.

$\therefore$ Variance $= E(X^2) - [E(X)]^2$

$= 6 - 2^2$

$= 6 - 4$

$$\boxed{\text{Variance} = 2}$$

(iii) $P(X \text{ is even}) = P(X=2) + P(X=4) + \cdots\cdots$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \cdots\cdots$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \cdots\cdots$$

$$= \frac{\left(\frac{1}{2}\right)^2}{1-\left(\frac{1}{2}\right)^2}$$

$\left[\because \text{It is a geometric series} = \frac{a}{1-r}\right]$

$a \to$ first term

$r \to$ common ratio

$$= \frac{1/4}{1-1/4}$$

$$= \frac{1}{3}$$

(iv) $P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + \cdots \infty$

$$= \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \cdots \infty$$

$$= \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^7 + \cdots \infty$$

$$= \frac{\left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}} = \frac{\frac{1}{32}}{\frac{1}{2}} = \frac{1}{16}$$

$$\boxed{P(X \geq 5) = \frac{1}{16}}$$

$P(X \text{ is divisible by } 3) = P(X=3) + P(X=6) + \cdots \infty$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots \infty$$

$$= \frac{\left(\frac{1}{2}\right)^3}{1 - \left(\frac{1}{2}\right)^3} = \frac{\frac{1}{8}}{\frac{7}{8}} = \frac{1}{7}$$

$$\boxed{P(X \text{ is divisible by } 3) = \frac{1}{7}}$$

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(3) Find the MGF for the distribution where

$$f(x) = \begin{cases} \frac{2}{3} & \text{at } x = 1 \\ \frac{1}{3} & \text{at } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution :

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} f(x)$$

$$= e^0 f(0) + e^t f(1) + e^{2t} f(2) + \cdots \infty$$

$$= 0 + e^t \cdot \frac{2}{3} + e^{2t}\left(\frac{1}{3}\right) + 0$$

$$\boxed{M_X(t) = \frac{2e^t}{3} + \frac{e^{2t}}{3}}$$

(4) Find the MGF of a random variable 'x' having the density function

$$f(x) = \begin{cases} \frac{x}{2} & , 0 \le x \le 2 \\ 0 & , \text{otherwise} \end{cases}$$

Using the generating fn find the first four moments about the origin.

Solution:

(i) $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$$= \int_0^2 e^{tx} \cdot \frac{x}{2}\, dx$$

$$= \frac{1}{2} \int_0^2 x e^{tx}\, dx$$

$u = x,\ v = e^{tx}$

$u' = 1,\ v_1 = \frac{e^{tx}}{t}$

$v_2 = \frac{e^{tx}}{t^2}$

$uv_1 - u'v_2$

$$= \frac{1}{2}\left[\frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{1}{t^2}\right]$$

$$\boxed{M_X(t) = \frac{1}{2t^2}\left[2te^{2t} - e^{2t} + 1\right]}$$

(ii) $M_X(t) = \frac{2te^{2t}}{2t^2} - \frac{e^{2t}}{2t^2} + \frac{1}{2t^2}$

$$= t^{-1} e^{2t} - \frac{1}{2} t^{-2} e^{2t} + \frac{1}{2} t^{-2}$$

$$M_X'(t) = e^{2t}\left(\frac{-1}{t^2}\right) + t^{-1} \cdot \frac{e^{2t}}{2} - \frac{1}{2}\left[\frac{e^{2t}}{2} t^{-2} + \frac{e^{2t}(-\dots}{t^3}\right. + \frac{1}{2}(-2)\frac{1}{t^3}$$

$$= -\frac{e^{2t}}{t^2} + \frac{e^{2t}}{2t} - \frac{e^{2t}}{4t^2} + \frac{e^{2t}}{t^3} - \frac{1}{t^3}$$

(5) Let 'X' be a random variable with p.d.f

$$f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Find (i) $P(X > 3)$ (ii) MGF of X (iii) $E(X)$ and $Var(X)$.

Solution:

Given : $f(x) = \begin{cases} 1/3 \, e^{-x/3} & , x > 0 \\ 0 & , \text{otherwise.} \end{cases}$

(i) $P(X > 3) = \int_3^\infty f(x)\, dx$

$$= \int_3^\infty \frac{1}{3} e^{-x/3}$$

$$= \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3} \right]_3^\infty$$

$$= e^{-1}$$

$$\boxed{P(X > 3) = 0.3679}$$

(ii) MGF of X :

$$M_X(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \frac{1}{3} e^{-x/3}\, dx$$

$$= \frac{1}{3} \int_0^\infty e^{(t - \frac{1}{3})x}\, dx = \frac{1}{3} \int_0^\infty e^{-(\frac{1}{3} - t)x}\, dx$$

$$= \frac{1}{3} \left[\frac{e^{-(\frac{1}{3} - t)x}}{-(\frac{1}{3} - t)} \right]_0^\infty$$

$$M_x(t) = -\frac{1}{3}\left[0 - \frac{1}{\frac{1}{3} - t}\right] = +\frac{1}{3}\left(\frac{1}{\frac{1-3t}{3}}\right)$$

$$\boxed{M_x(t) = \frac{1}{1-3t}}$$

(iii) $M_x(t) = \frac{1}{1-3t} = (1-3t)^{-1}$

$$M_x'(t) = -\frac{1}{(1-3t)^2}(-3) = \frac{3}{(1-3t)^2}$$

$$M_x'(0) = \frac{3}{1-0} = 3$$

$$\boxed{E(X) = \text{Mean} = M_x'(0) = 3}$$

$$M_x''(t) = -6(1-3t)^{-3}(-3)$$

$$= 18(1-3t)^{-3}$$

$$M_x''(0) = 18$$

$$E(X^2) = M_x''(0) = 18$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 18 - (3)^2$$

$$= 18 - 9$$

$$\boxed{Var(X) = 9}$$