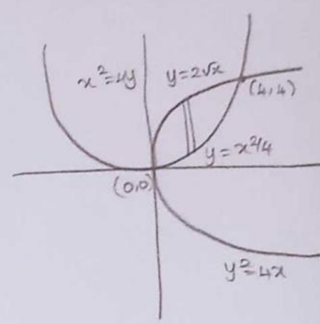
UNIT 5 – Multiple Integrals

Problems based on Area as a double Integral in Cartesian coordinates:

1. Find the area between enclosed by the curves $y^2 = 4x$ and $x^2 = 4y$



Given that

$y^2 = 4x \rightarrow ①$

$x^2 = 4y$

$y = \frac{x^2}{4} \rightarrow ②$

Sub y in ①

$\left(\frac{x^2}{4}\right)^2 = 4x$

$\frac{x^4}{16} = 4x$

$x^3 = 64$

$\boxed{x = 4}$

dydx → Vertical Strip
y limit → interms of x
x limit → Constant limits

Subs $x = 4$ in ②

$y = \frac{x^2}{4}$

$= \frac{16}{4}$

$\boxed{y = 4}$

Intersection points is $(4,4)$

Limits:

x : 0 to 4

y : $\frac{x^2}{4}$ to $2\sqrt{x}$

We know that

$$A = \iint_R dy\,dx = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\,dx$$

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$$= \int_0^4 [y]_{y=x^2/4}^{2\sqrt{x}} dx = \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$= \int_0^4 \left[2x^{1/2} - \frac{x^2}{4}\right] dx = \left[\frac{2x^{3/2}}{3/2} - \frac{x^3}{4\times 3}\right]_0^4$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 = \left[\left(\frac{4}{3}(4)^{3/2} - \frac{(4)^3}{12}\right) - 0\right]$$

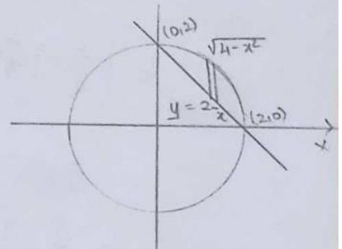
$$= \left[\frac{4}{3}.(8) - \frac{64}{12}\right] = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}.$$

2. Find the smallest area bounded by $y = 2-x$, $x^2+y^2 = 4$

Solution:-

Given that $y = 2-x \rightarrow ①$

$x^2+y^2 = 4 \rightarrow ②$



sub ① in ②

$x^2 + (2-x)^2 = 4$

$x^2 + 4 + x^2 - 4x = 4$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

$x = 0, x = 2$

dydx → Vertical strip
y limit → interms of x
x limit → Constant limits

$x^2+y^2 = 4$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4-x^2}$

When $x = 0$, $y = 2-0 = 2$

$x = 2$, $y = 2-2 = 0$

∴ The intersection points are $(0,2)$ & $(2,0)$

Limits : x : 0 to 2

y : $2-x$ to $\sqrt{4-x^2}$

$$Area = \iint_R dy\,dx = \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy\,dx$$

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$$= \int_0^2 [y]_{2-x}^{\sqrt{4-x^2}}\, dx$$

$$= \int_0^2 \left[\sqrt{4-x^2} - (2-x)\right] dx$$

$$= \int_0^2 \sqrt{4-x^2}\, dx - \int (2-x)\, dx$$

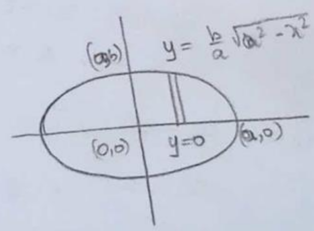
$$= \left[\frac{x}{2}\sqrt{2^2-x^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[\frac{2}{2}\sqrt{2^2-2^2} + 2\sin^{-1}\left(\frac{2}{2}\right)\right] - 0 - \left[4 - \frac{4}{2}\right]$$

$$= 2\sin^{-1}(1) - [4-2]$$

$$= 2\left(\frac{\pi}{2}\right) - 2 \quad \boxed{A = \pi - 2}$$

2. Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ using double integration



Given that:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$= \frac{a^2-x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

$dy\,dx \rightarrow$ Vertical strip

y limit $\rightarrow$ interms of x

x limit $\rightarrow$ constant limit

Limits

x : 0 to a

y : 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

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The area of the ellipse is

$A = 4 \times$ Area in the first quadrant

$$= 4 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$$

$$= 4 \int_0^a \left[y \right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4 \int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2} - 0 \right] dx$$

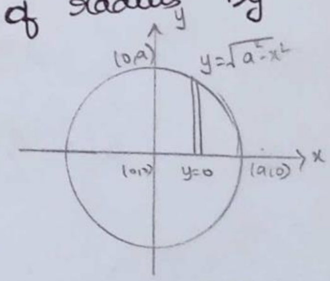
$$= \frac{4b}{a} \int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \frac{4b}{a} \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a$$

$$= \frac{4b}{a} \left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a} \right]$$

$$= \frac{4b}{a} \left[\frac{a^2}{2}\left(\frac{\pi}{2}\right) \right] = \pi ab$$

4. Find the area of a circle of radius by double integration



Equation of the circle is

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

$$y = \pm\sqrt{a^2-x^2}$$

dy dx → vertical strip

y limit → interms of x

x limit → Constant limits

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Limits:

x : 0 to a

y : 0 to $\sqrt{a^2-x^2}$

The area of the circle is

A = 4 × area in the 1st quadrant

$$= 4\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx$$

$$= 4\int_0^a \left[y\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a \left[\sqrt{a^2-x^2} - 0\right] dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left[0 + \frac{a^2}{2}\sin^{-1} 1\right] - 0\right]$$

$$= 2a^2\left[\frac{\pi}{2}\right]$$

$$A = \pi a^2.$$