



UNIT 5 - Multiple Integrals

Double integration under a given region

Standard diagrams:

Equation	Graph.
1. $y = x$	
2. $x + y = 1$	
3. $x^2 + y^2 = a^2$	
4. $y^2 = 4ax$	
5. $x^2 = 4ay$	
6. $x = a, y = b$	



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UNIT 5 – Multiple Integrals

Double integration, Based on region

$\iint dx dy \rightarrow$ horizontal strip [left to right] $\begin{matrix} x - \text{variable of } y \\ y - \text{constant limit} \end{matrix}$

$\iint dy dx \rightarrow$ vertical strip [down to up] $\begin{matrix} x \rightarrow \text{constant limit} \\ y \rightarrow \text{variable of } x \end{matrix}$

1. Evaluate $\iint_R xy \, dx dy$ where R is the 1st quadrant of the circle $x^2 + y^2 = a^2$

Given : $x^2 + y^2 = a^2$
 $x^2 = a^2 - y^2$
 $x = \pm \sqrt{a^2 - y^2}$

At y axis,
take $x=0$, $y^2 = a^2$
 $y = \pm a$

Limits $x : 0$ to $\sqrt{a^2 - y^2}$
 $y : 0$ to a

Now, $\iint_R xy \, dx dy = \int_0^a \int_0^{\sqrt{a^2 - y^2}} xy \, dx dy$

$= \int_0^a \left[\frac{x^2 y}{2} \right]_0^{\sqrt{a^2 - y^2}} dy = \int_0^a \left[\frac{(a^2 - y^2)y}{2} - 0 \right] dy$

$= \frac{1}{2} \int_0^a (a^2 y - y^3) dy = \frac{1}{2} \left[\frac{a^2 y^2}{2} - \frac{y^4}{4} \right]_0^a$

$= \frac{1}{2} \left[\left(\frac{a^4}{2} - \frac{a^4}{4} \right) - 0 \right] = \frac{a^4}{2} \left[\frac{1}{2} - \frac{1}{4} \right]$

$= \frac{a^4}{2} \left(\frac{1}{4} \right)$

$\iint_R xy \, dx dy = \frac{a^4}{8}$

$x^2 + y^2 = a^2$
 $dx dy =$ horizontal strip
 x limit - intervals of y
 y limit - constant limits



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2. Evaluate $\iint_R xy(x+y) dy dx$ over the area between

$$y = x^2 \text{ and } y = x$$

Given: $y = x^2$ and $y = x$.

$$\therefore x^2 = x$$

$$x^2 - x = 0 \Rightarrow x(x-1) = 0$$

$$x = 0 \text{ and } x = 1$$

When $x = 0$, $y = 0$

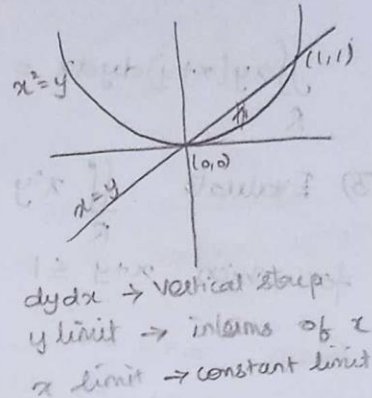
When $x = 1$, $y = 1$

Intersection point $(0,0)$ & $(1,1)$

Limits: $x : 0$ to 1

$y : x^2$ to x

$$\begin{aligned} \iint_R xy(x+y) dy dx &= \int_0^1 \int_{x^2}^x xy(x+y) dy dx \\ &= \int_0^1 \int_{x^2}^x (x^2y + xy^2) dy dx \\ &= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{y=x^2}^x dx \\ &= \int_0^1 \left[\left(\frac{x^4}{2} + \frac{x^4}{3} \right) - \left(\frac{x^6}{2} + \frac{x^7}{3} \right) \right] dx \\ &= \int_0^1 \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx \\ &= \left[\frac{5}{6} \cdot \frac{x^5}{5} - \frac{1}{2} \frac{x^7}{7} - \frac{1}{3} \frac{x^8}{8} \right]_0^1 \\ &= \left[\frac{1}{6}(1) - \frac{1}{14}(1) - \frac{1}{24}(1) \right] = 0 \\ &= \frac{4-1}{24} - \frac{1}{14} = \frac{3}{24} - \frac{1}{14} = \frac{1}{8} - \frac{1}{14} = \frac{42-24}{336} \end{aligned}$$





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$$= \frac{18}{336}$$

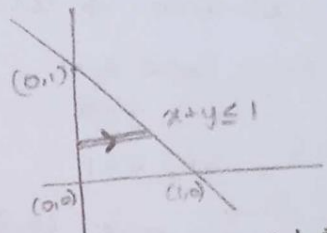
$$\iint_R xy(x+y) dy dx = \frac{3}{56}$$

3) Evaluate $\iint_R x^2y dx dy$ over the region in the +ve quadrant for which $x+y \leq 1$

Given $x+y=1$
 $y=0, x=1 \Rightarrow (1,0)$
 $x=0, y=1 \Rightarrow (0,1)$

Intersection points $(1,0)$ & $(0,1)$

Limits: $x: 0$ to $1-y$
 $y: 0$ to 1



$dxdy \rightarrow$ horizontal strip
 x limit \rightarrow intervals of y
 y limit \rightarrow constant limits

$$\begin{aligned} \iint_R x^2y dx dy &= \int_0^1 \int_0^{1-y} x^2y dx dy = \int_0^1 \left[\frac{x^3}{3} y \right]_{x=0}^{1-y} dy \\ &= \int_0^1 \left[\frac{(1-y)^3}{3} y - 0 \right] dy = \frac{1}{3} \int_0^1 y(1-3y+3y^2-y^3) dy \\ &= \frac{1}{3} \int_0^1 [y - 3y^2 + 3y^3 - y^4] dy \\ &= \frac{1}{3} \left[\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5} \right]_0^1 \\ &= \frac{1}{3} \left[\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right] = \frac{1}{3} \left[\frac{20-40+30-8}{40} \right] \\ &= \frac{1}{3} \left[\frac{2}{40} \right] \end{aligned}$$

$$\iint_R x^2y dx dy = \frac{1}{60}$$



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4. Evaluate $\iint_R x^2 y \, dy \, dx$ lies on the 1st quadrant of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Given that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$
$$y^2 = b^2 \left(\frac{a^2 - x^2}{a^2} \right)$$
$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

At x axis

$$y=0 \Rightarrow \frac{x^2}{a^2} + 0 = 1$$
$$x^2 = a^2$$
$$x = \pm a$$

Limits : $x=0$ to a
 $y: 0$ to $\frac{b}{a} \sqrt{a^2 - x^2}$

$dy \, dx \rightarrow$ Vertical strip
 y limit \rightarrow intervals of x
 x limit \rightarrow constant limit

$$\iint_R x^2 y \, dy \, dx = \int_0^a \int_0^{\frac{b}{a} \sqrt{a^2 - x^2}} x^2 y \, dy \, dx$$
$$= \int_0^a x^2 \left[\frac{y^2}{2} \right]_{y=0}^{\frac{b}{a} \sqrt{a^2 - x^2}} dx$$
$$= \int_0^a \frac{x^2}{2} \left[\frac{b^2}{a^2} (a^2 - x^2) - 0 \right] dx$$
$$= \frac{b^2}{2a^2} \int_0^a x^2 (a^2 - x^2) dx$$
$$= \frac{b^2}{2a^2} \int_0^a (a^2 x^2 - x^4) dx$$



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$$= \frac{b^2}{2a^2} \left[\frac{a^2 x^3}{3} - \frac{x^5}{5} \right]_0^a$$

$$= \frac{b^2}{2a^2} \left[\left(\frac{a^5}{3} - \frac{a^5}{5} \right) - 0 \right]$$

$$= \frac{b^2 a^5}{2a^2} \left[\frac{1}{3} - \frac{1}{5} \right]$$

$$= \frac{b^2 a^3}{2} \left[\frac{5-3}{15} \right] = \frac{b^2 a^3}{2} \left[\frac{2}{15} \right]$$

$$= \frac{a^3 b^2}{15}$$

$$\iint_R x^2 y \, dy \, dx = \frac{a^3 b^2}{15}$$

(1) Using double integration find the area enclosed by the curves
 $y = 2x^2$ & $y^2 = 4x$.

$y = 2x^2 \rightarrow$ ①
 $y^2 = 4x \rightarrow$ ②

Sub ① in ②
 $(2x^2)^2 = 4x$
 $4x^4 = 4x$
 $x^4 - x = 0 \quad x(x^3 - 1) = 0$
 $x = 0, \quad x^3 = 1 \Rightarrow x = 1$

When $x = 0, \quad y = 0$
 $x = 1, \quad y = 2$

$x: 0 \text{ to } 1$
 $y: 2x^2 \text{ to } 2\sqrt{x}$

$$A = \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy \, dx$$

$$= \int_0^1 [2\sqrt{x} - 2x^2] dx$$

$$= 2 \left[\frac{2x^{3/2}}{3/2} - \frac{2x^3}{3} \right]_0^1$$

$$= \frac{8}{3}$$