UNIT5–MultipleIntegrals

Triple Integration in Cartesian Coordinates

Triple integration of a function defined over a region is

$$\iiint_R f(x,y,z)\,dx\,dy\,dz$$

Note:

$$\iiint_R dx\,dy\,dz \rightarrow \text{Volume of the region } R.$$

Problems:

1. Evaluate $\int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx$.

Solution:

$$\int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx = \int_0^2\int_1^3 xy^2\left(\frac{z^2}{2}\right)_1^2 dy\,dx$$

$$= \int_0^2\int_1^3 xy^2\left[\frac{4}{2}-\frac{1}{2}\right]dy\,dx$$

$$= \frac{3}{2}\int_0^2\int_1^3 xy^2\,dy\,dx = \frac{3}{2}\int_0^2 x\left[\frac{y^3}{3}\right]_1^3 dx$$

$$= \frac{1}{2}\int_0^2 x\left[27-1\right]dx$$

$$= \frac{26}{2}\int_0^2 x\,dx = 13\left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{13}{2}\left[4\right] = 26.$$

$$\int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx = 26.$$

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2. Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$ $\qquad \int_0^a\int_0^b\int_0^c e^{x+y+z}\,dx\,dy\,dz$

Ans: $(e^c-1)(e^b-1)(e^a-1)$

Solution:

$$\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 e^x e^y e^z\,dx\,dy\,dz$$

$$= \int_0^1 e^x\,dx \int_0^1 e^y\,dy \int_0^1 e^z\,dz$$

$$= \left[e^x\right]_0^1 \left[e^y\right]_0^1 \left[e^z\right]_0^1$$

$$= (e^1-e^0)(e^1-e^0)(e^1-e^0)$$

$$= (e^1-e^0)^3$$

$$= 5.0732$$

3. Evaluate $\int_0^a\int_0^b\int_0^c xyz\,dx\,dy\,dz$ $\quad a=1,\ b=2,\ c=3$ $\quad = 9/2$

Solution:

$$\int_0^a\int_0^b\int_0^c xyz\,dx\,dy\,dz = \int_0^a\int_0^b \left(\frac{x^2}{2}\right)_0^c yz\,dy\,dz$$

$$= \frac{c^2}{2}\int_0^a\int_0^b yz\,dy\,dz$$

$$= \frac{c^2}{2}\int_0^a \left(\frac{y^2}{2}\right)_0^b z\,dz$$

$$= \frac{b^2c^2}{4}\int_0^a z\,dz$$

$$= \frac{b^2c^2}{4}\left[\frac{z^2}{2}\right]_0^a = \frac{b^2c^2}{4}\left[\frac{a^2}{2}\right]$$

$$\int_0^a\int_0^b\int_0^c xyz\,dx\,dy\,dz = \frac{a^2b^2c^2}{8} \quad \frac{1\cdot 4\cdot 9}{8} = 9/2$$

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4) Evaluate $\int_0^1\int_0^y\int_0^{x+y} dx\,dy\,dz$

Solution:

$$\int_0^1\int_0^y\int_0^{x+y} dz\,dx\,dy \text{ (correct form)} = \int_0^1\int_0^y [z]_0^{x+y}\,dx\,dy = \int_0^1\int_0^y [x+y-0]\,dx\,dy$$

$$= \int_0^1\int_0^y (x+y)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2}+xy\right]_0^y dy$$

$$= \int_0^1 \left[\left(\frac{y^2}{2}+y^2\right)-0\right] dy$$

$$= \frac{3}{2}\int_0^1 y^2\,dy = \frac{3}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$\int_0^1\int_0^y\int_0^{x+y} dx\,dy\,dz = \frac{1}{2}$$

5. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

Solution:

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} xy\left[\frac{1-x^2-y^2}{2}-0\right] dy\,dx.$$

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$$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} \left[xy - x^3y - xy^3\right] dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x(1-x^2)}{2} - \frac{x^3[1-x^2]}{2} - \frac{x[1-x^2]^2}{4}\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x-x^3}{2} - \frac{x^3-x^5}{2} - \frac{x[1-2x^2+x^4]}{4}\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x-x^3-x^3+x^5}{2} - \frac{x}{4} + \frac{2x^3}{4} - \frac{x^5}{4}\right] dx$$

$$= \frac{1}{2(4)}\int_0^1 \left[2(x-2x^3+x^5) - x - x^5 + 2x^3\right] dx$$

$$= \frac{1}{8}\int_0^1 (2x - 4x^3 + 2x^5 - x - x^5 + 2x^3)\, dx$$

$$= \frac{1}{8}\int_0^1 (x - 2x^3 + x^5)\, dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{1}{48}$$

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz = \frac{1}{48}$$