SNS COLLEGE OF TECHNOLOGY
COIMBATORE

Department of Civil Engineering

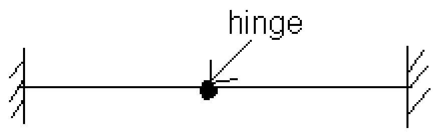
QUESTION BANK

19CET306 – STRUCTURAL ANALYSIS II

UNIT 1- FLEXIBILITY METHOD

<u>PART A - 2 mark Questions and Answers</u>

1. Find degree of indeterminacy of the following.



Degree of indeterminacy = No. of reactions – No. of condition equations
= (3 + 2 + 3) – 3
= 5

2. Define kinematic redundancy.

When a structure is subjected to loads, each joint will undergo displacements in the form of translations and rotations. Kinematic redundancy of a structure means the number of unknown joint displacement in a structure.

3. Give the mathematical expression for the degree of static indeterminacy of rigid jointed plane frames.

Degree of static indeterminacy = (No. of closed loops x 3) – No. of releases

4. What are the properties which characterize the structure response by means of force-displacement relationship?

- Each element of a flexibility matrix represents a displacement at a coordinate (i) due to a force at a coordinate (j).
- If the matrix of the structure is known, we know the behaviour of the structure.

5. What are the conditions to be satisfied for determinate structures and how are indeterminate structures identified?

Determinate structures can be solving using conditions of equilibrium alone (H = 0; V = 0; M = 0). No other conditions are required.

Indeterminate structures cannot be solved using conditions of equilibrium because ($H \neq 0$; $V \neq 0$; $M \neq 0$). Additional conditions are required for solving such structures.

6. Write down the equation for the degree of static indeterminacy of the pin-jointed frames, explaining the notations used.

Total indeterminacy = External indeterminacy + Internal indeterminacy External indeterminacy = No. of reactions – No. of equilibrium equations

Internal indeterminacy m = 2 j-3⁻

7. Differentiate pin-jointed plane frame and rigid jointed plane frame.

S.No	Pin jointed plane frame	Rigid jointed plane frame
1	The joints permit change of angle between connected members.	The members connected at a rigid joint with maintain the angle between them even under deformation due to loads.
2	The joints are incapable of transferring any moment to the connected members and vice-versa.	Members can transmit both forces and moments between themselves through the joint.
3	The pins transmit forces between connected members by developing shear.	Provision of rigid joints normally increases the redundancy of the structures.

8. Mention any two methods of determining the joint deflection of a perfect frame.

- Unit load method
- Virtual work method
- Slope deflection method
- Strain energy method

9. What are the requirements to be satisfied while analyzing a structure?

The three conditions to be satisfied are:

(i) Equilibrium condition

(ii) Compatibility condition

(iii) Force displacement condition

10. What is meant by force method in structural analysis?

A method in which the forces are treated as unknowns is known as force method.

The following are the force methods:

- Flexibility matrix method
- Consistent deformation method
- Claypeyron's 3 moment method
- Column analogy method

11. Define flexibility coefficient.

It is defined as the displacement at coordinate i due to unit force at coordinate j in a structure. It makeup the elements of a flexibility matrix.

12. Why is flexibility method also called as compatibility method or force method?

Flexibility method begins with the superposition of forces and is hence known as force method. Flexibility method leads to equations of displacement compatibility and is hence known as compatibility method.

13. Define the Force Transformation Matrix.

The connectivity matrix which relates the internal forces Q and the external forces R is known as the force transformation matrix. Writing it in a matrix form,

$$\{Q\} = [b]\ \{R\}$$

14. State any two methods of matrix inversion.

- Adjoint method
- The gauss-jordan method (by linear transformation)
- The Choleski method (by factorization)
- Partitioning method

15. Define Degree of Freedom and explain its types.

Degree of freedom is defined as the least no of independent displacements required to define the deformed shape of a structure.

There are two types of DOF: (a) Nodal type DOF and (b) Joint type DOF.

a) Nodal type DOF:

This includes the DOF at the point of application of concentrated load or moment, at a section where moment of inertia changes, hinge support, roller support and junction of two or more members.

b) Joint type DOF:

This includes the DOF at the point where moment of inertia changes, hinge and rollersupport and junction of two or more members.

16. Define a primary structure.

A structure formed by the removing the excess or redundant restraints from an indeterminate structure making it statically determinate is called primary structure. This is required for solving indeterminate structures by flexibility matrix method.

17. Briefly mention the two types of matrix methods of analysis of indeterminate structures.

Flexibility matrix method:

This method is also called the force method in which the forces in the structure are treated as unknowns. The no of equations involved is equal to the degree of static indeterminacy of the structure.

Stiffness matrix method:

This is also called the displacement method in which the displacements that occur in the structure are treated as unknowns. The no of displacements involved is equal to the no of degrees of freedom of the structure.

19. Define local and global coordinates.

Local coordinates:

Coordinates defined along the individual member axes locally.

Global coordinates:

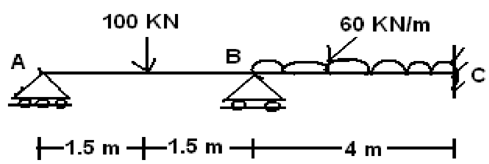
Common coordinate system dealing with the entire structure. Also known as system coordinates.

20. What is the relation between the flexibility matrix and stiffness matrix?

The relation between the flexibility matrix and stiffness matrix is that, one is the inverse of the other, when they both exist.

Part B – 16 Marks Questions and Answers

1. Analyse the continuous beam shown in figure using force method. (AUC Apr/May 2011)



Solution:

Step1: Static Indeterminacy :

Degree of redundancy = (1 + 1 + 3) - 3 = 2

Release at B and C by apply hinge.

Step 2: Fixed End Moment :

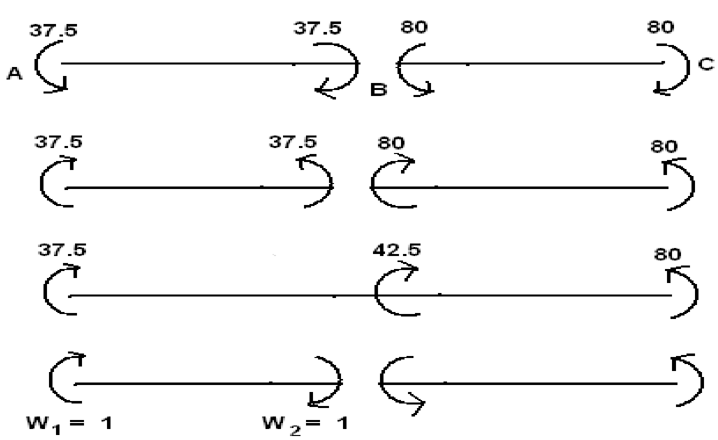
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{100 \times 3}{8} = -37.5 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{100 \times 3}{8} = 37.5 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{60 \times 4^2}{12} = -80 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell^2}{12} = \frac{60 \times 4^2}{12} = 80 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B):

$$B = B_W B_X$$

$$B_W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}$$

$$F_X = B_x^T \, F \, B_x$$

$$= \frac{1}{EI} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 2.33 & -0.67 \\ -0.67 & 1.33 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.502 & 0.253 \\ 0.253 & 0.879 \end{bmatrix}$$

$$F_W = B_x^T \, F \, B_w$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} 0.5 & -1 \\ 0 & 0 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} \, F_W \, W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.502 & 0.253 \\ 0.253 & 0.879 \end{bmatrix}\begin{bmatrix} 0.5 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 37.5 \\ 42.5 \end{bmatrix}$$

$$= -\begin{bmatrix} 0.251 & 0.502 \\ 0.127 & -0.253 \end{bmatrix}\begin{bmatrix} 37.5 \\ 42.5 \end{bmatrix}$$

$$= -\begin{bmatrix} -11.923 \\ -5.99 \end{bmatrix}$$

$$X = \begin{bmatrix} 11.923 \\ 5.99 \end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 37.5 \\ 42.5 \\ 11.923 \\ 5.99 \end{bmatrix}$$

$$P = \begin{bmatrix} 37.5 \\ 30.58 \\ 11.923 \\ 5.99 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -37.5 \\ 37.5 \\ -80 \\ 80 \end{bmatrix} + \begin{bmatrix} 37.5 \\ 30.58 \\ 11.923 \\ 5.99 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 \\ 68.08 \\ -68.08 \\ 95.99 \end{bmatrix}$$

2. Analyse the portal frame ABCD shown in figure using force method. (AUC Apr/May 2011)

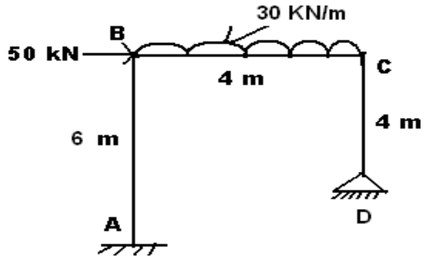


Fig.

Solution:

Step1: Static Indeterminacy :

Degree of redundancy = (3 + 2) - 3 = 2

Release at B and C by apply hinge.

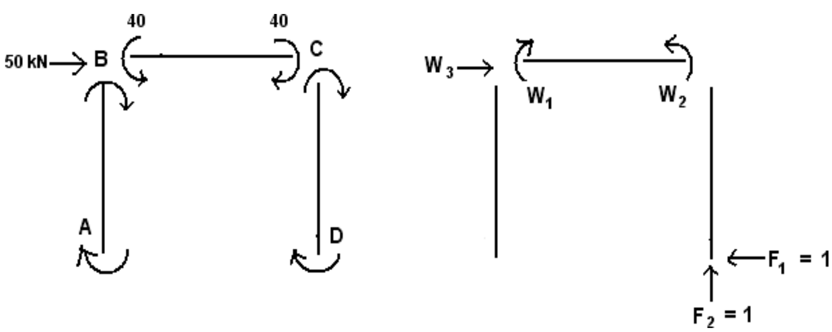
Apply a unit force at B joint.

Step 2: Fixed End Moment :

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{30 \times 4^2}{12} = -\ 40 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell^2}{12} = \frac{30 \times 4^2}{12} = 40 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co efficient matrix (B):

$$B = B_W B_X$$

$$B_W = \begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} -2 & 4 \\ 4 & 4 \\ -4 & 4 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & -6 & -2 & 4 \\ 0 & 0 & 0 & 4 & 4 \\ 1 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 & 0 & 0 \\ 0 & 0 & 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0 & 0 & -0.67 & 1.33 \end{bmatrix}$$

$$F_X = B_x^T \; F \; B_x$$

$$= \frac{1}{EI}\begin{bmatrix} -2 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} -8 & 10 & -8 & 8 & -5.32 & 2.68 \\ 12 & -12 & 5.32 & -2.68 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI}\begin{bmatrix} 141.28 & -104 \\ -104 & 117.28 \end{bmatrix}$$

$$F_x^{-1} = EI\begin{bmatrix} 0.0204 & 0.0181 \\ 0.0181 & 0.0246 \end{bmatrix}$$

$$F_W = B_x^T\, F\, B_w$$

$$= \frac{1}{EI}\begin{bmatrix} -2 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} -8 & 10 & -8 & 8 & -5.32 & 2.68 \\ 12 & -12 & 5.32 & -2.68 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} -8 & -5.32 & 48 \\ 5.32 & 0 & -72 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} F_W W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.0204 & 0.0181 \\ 0.0181 & 0.0246 \end{bmatrix}\begin{bmatrix} -8 & -5.32 & 48 \\ 5.32 & 0 & -72 \end{bmatrix}\begin{bmatrix} 40 \\ -40 \\ 50 \end{bmatrix}$$

$$= -\begin{bmatrix} 0.0669 & 0.1085 & 0.3240 \\ 0.0139 & 0.0963 & 0.9024 \end{bmatrix}\begin{bmatrix} 40 \\ -40 \\ 50 \end{bmatrix}$$

$$= -\begin{bmatrix} -14.536 \\ -41.824 \end{bmatrix}$$

$$X = \begin{bmatrix} 14.536 \\ 41.824 \end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 & -2 & 4 \\ 0 & 0 & 0 & 4 & -4 \\ 1 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 40 \\ -40 \\ 50 \\ 14.536 \\ 41.824 \end{bmatrix}$$

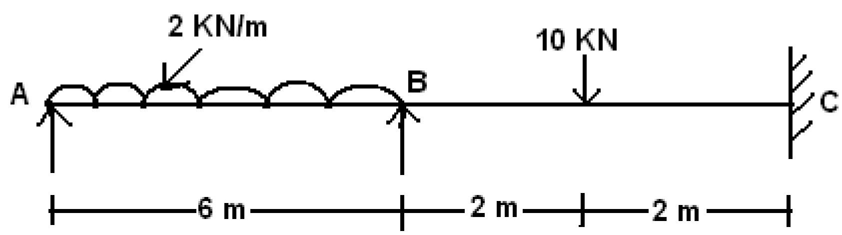
$$P = \begin{bmatrix} -161.776 \\ -109.152 \\ 149.152 \\ 58.144 \\ -98.144 \\ 0 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -40 \\ 40 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -161.776 \\ -109.152 \\ 149.152 \\ 58.144 \\ -98.144 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} -161.776 \\ -109.152 \\ 109.152 \\ 98.144 \\ -98.144 \\ 0 \end{bmatrix}$$

3. Analyse the continuous beam ABC shown in figure by flexibility matrix method and sketch the bending moment diagram. (AUC Nov/Dec 2011).



EI = Constant

Solution:

Step1: Static Indeterminacy :

Degree of redundancy = (1 + 1 + 3) - 3 = 2

Release at B and C by apply hinge.

Step 2: Fixed End Moment :

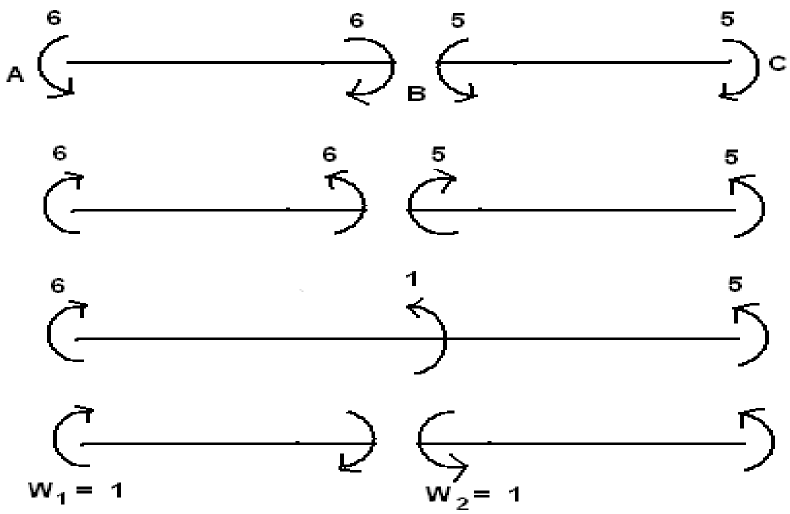
$$M_{FAB} = -\frac{w\ell^2}{12} = -\frac{2 \text{ x } 6^2}{12} = -6 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell^2}{12} = \frac{2 \text{ x } 6^2}{12} = 6 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{10 \text{ x } 4}{8} = -5 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell}{8} = \frac{10 \text{ x } 4}{8} = 5 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B):

$$B = B_W B_X$$

$$B_W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}$$

$$F_X = B_x^T \, F \, B_x$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 1 & -2 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_x = \frac{1}{EI}\begin{bmatrix} 3.33 & -0.67 \\ -0.67 & 1.33 \end{bmatrix}$$

$$F_x^{-1} = EI\begin{bmatrix} 0.334 & 0.168 \\ 0.168 & 0.837 \end{bmatrix}$$

$$F_W = B_x^T \; F \; B_w$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 1 & -2 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} 1 & 1.33 \\ 0 & -0.67 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} \; F_W \; W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.334 & 0.168 \\ 0.168 & 0.837 \end{bmatrix}\begin{bmatrix} 1 & 1.33 \\ 0 & -0.67 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$= -\begin{bmatrix} 0.334 & 0.3316 \\ 0.168 & -0.337 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$= -\begin{bmatrix} 1.672 \\ 1.345 \end{bmatrix}$$

$$X = \begin{bmatrix} -1.672 \\ -1.345 \end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \\ -1.672 \\ -1.345 \end{bmatrix}$$

$$P = \begin{bmatrix} 6 \\ 1.672 \\ -2.672 \\ -1.345 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -6 \\ 6 \\ -5 \\ 5 \end{bmatrix} + \begin{bmatrix} 6 \\ 1.672 \\ -2.672 \\ -1.345 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 \\ 7.672 \\ -7.672 \\ 3.655 \end{bmatrix}$$

4. Analyse the portal frame ABCD shown in figure by flexibility matrix method and sketch the bending moment diagram. (AUC Nov/Dec 2011)

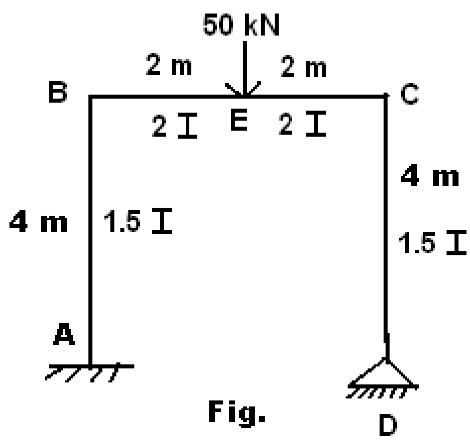


Fig.

Solution:

Step1: Static Indeterminacy :

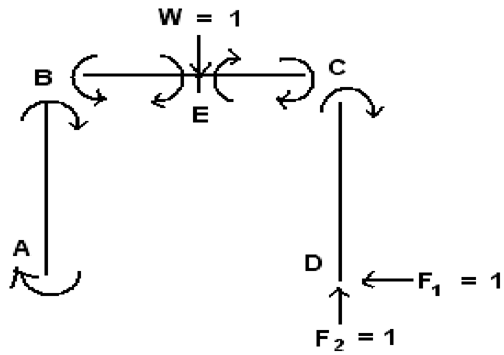
Degree of redundancy = (3 + 2) - 3 = 2

Release at D by apply horizontal and vertical supports.

Step 2: Fixed End Moment :

$M_{FAB} = M_{FBA} = M_{FBC} = M_{FBC} = M_{FCD} = M_{FDC} = 0$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B):

$B = B_W B_X$

$$B_W = \begin{bmatrix} -2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 0 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & -2 \\ -4 & 2 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 & 4 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \\ 0 & 4 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F) :

$$F = \frac{L}{6EI}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI}\begin{bmatrix} 0.89 & -0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.44 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & -0.17 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.17 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & 0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.17 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & 0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.89 \end{bmatrix}$$

$$F_X = B_x^T \, F \, B_x$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & 4 & -4 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & -2 & 2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.89 & 0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.44 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & 0.17 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.17 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & 0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.17 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & 0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.89 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & -2 \\ -4 & 2 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} -1.76 & 3.56 & -2 & 2 & -2 & 2 & -3.56 & 1.76 \\ 5.32 & -5.32 & 1.66 & -1.34 & 0.66 & -0.34 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & -2 \\ -4 & 2 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI}\begin{bmatrix} 60.48 & -37.28 \\ -37.28 & 53.2 \end{bmatrix}$$

$$F_x^{-1} = EI\begin{bmatrix} 0.0291 & 0.0203 \\ 0.0203 & 0.033 \end{bmatrix}$$

$$F_W = B_x^T \; F \; B_w$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & 4 & -4 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & -2 & 2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.89 & -0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.44 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & 0.17 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.17 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & 0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.17 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & 0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.89 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} -1.76 & 3.56 & -2 & 2 & -2 & 2 & -3.56 & 1.76 \\ 5.32 & -5.32 & 1.66 & -1.34 & 0.66 & -0.34 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} 14.64 \\ -24.60 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} F_W W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.0291 & 0.0203 \\ 0.0203 & 0.033 \end{bmatrix}\begin{bmatrix} 14.64 \\ -24.60 \end{bmatrix} 50$$

$$= -\begin{bmatrix} -0.0734 \\ -0.5146 \end{bmatrix} 50$$

$$= -\begin{bmatrix} -3.67 \\ -25.73 \end{bmatrix}$$

$$X = \begin{bmatrix} 3.67 \\ 25.73 \end{bmatrix}$$

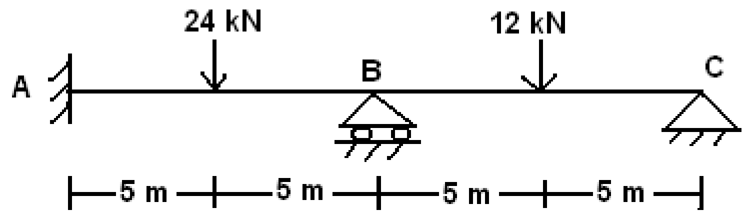
Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \\ 0 & 4 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 50 \\ 3.67 \\ 25.73 \end{bmatrix}$$

$$P = \begin{bmatrix} 2.92 \\ 11.76 \\ -11.76 \\ -36.78 \\ 36.78 \\ 14.68 \\ -14.68 \\ 0 \end{bmatrix}$$

The final moments also same, since there are no external forces acting on the members.

5. **Analyse the continuous beam ABC shown in figure by flexibility matrix method and sketch the bending moment diagram.** **(AUC May/June 2012)**



Solution:

Step1: Static Indeterminacy :

Degree of redundancy = (3 + 1 + 1) - 3 = 2

Release at A and B by apply hinge.

Step 2: Fixed End Moment :

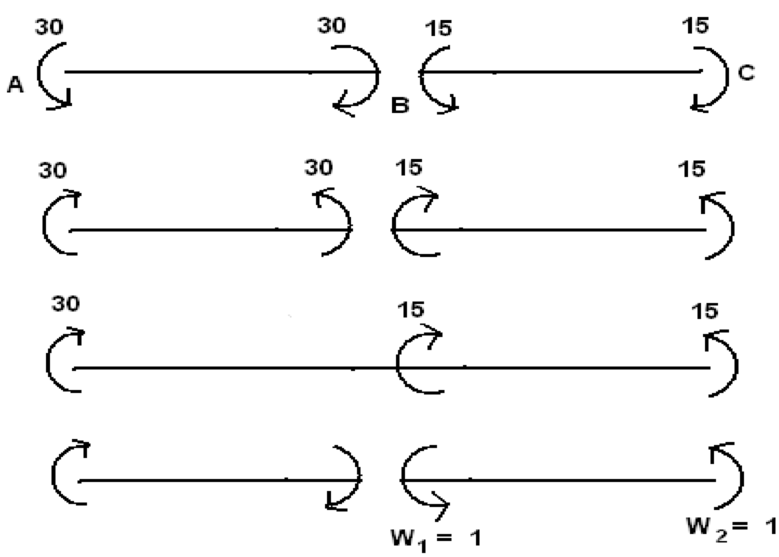
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{24 \text{ x } 10}{8} = -30 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{24 \text{ x } 10}{8} = 30 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{12 \text{ x } 10}{8} = -15 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell}{8} = \frac{12 \text{ x } 10}{8} = 15 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B) :

$$B = B_W \, B_X$$

$$B_W = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Step5: Flexibility matrix (F) :

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 3.33 & 1.67 & 0 & 0 \\ 1.67 & 3.33 & 0 & 0 \\ 0 & 0 & 3.33 & 1.67 \\ 0 & 0 & 1.67 & 3.33 \end{bmatrix}$$

$$F_X = B_x^T \, F \, B_x$$

$$= \frac{1}{EI} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ -1.67 & 3.33 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 3.33 & 1.67 \\ 1.67 & 6.66 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.3435 & -0.086 \\ -0.086 & 0.1717 \end{bmatrix}$$

$$F_W = B_x^T \, F \, B_w$$

$$= \frac{1}{EI}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ -1.67 & 3.33 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} \, F_W \, W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.3435 & -0.086 \\ -0.086 & 0.1717 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix}\begin{bmatrix} -15 \\ -15 \end{bmatrix}$$

$$= -\begin{bmatrix} -0.286 & 0.144 \\ 0.144 & -0.286 \end{bmatrix}\begin{bmatrix} -15 \\ -15 \end{bmatrix}$$

$$= -\begin{bmatrix} 2.13 \\ -4.29 \end{bmatrix}$$

$$X = \begin{bmatrix} 2.13 \\ 4.29 \end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -15 \\ -15 \\ -2.13 \\ 4.29 \end{bmatrix}$$

$$P = \begin{bmatrix} -2.13 \\ -4.29 \\ -10.71 \\ -15 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -30 \\ 30 \\ -15 \\ 15 \end{bmatrix} + \begin{bmatrix} -2.13 \\ -4.29 \\ -10.71 \\ -15 \end{bmatrix}$$

$$M = \begin{bmatrix} -32.13 \\ 25.71 \\ -25.71 \\ 0 \end{bmatrix}$$

UNIT 2 - STIFFNESS MATRIX METHOD

Part A – 2 Mark Questions and Answers

1. **Define static indeterminacy.**

   The excess number of reactions that make a structure indeterminate is called staticindeterminacy.
   Static indeterminacy = No. of reactions – Equilibrium conditions

2. **Define flexibility of a structure.**

   This method is also called the force method in which the forces in the structure are treatedas unknowns. The no of equations involved is equal to the degree of static indeterminacy of the structure.

3. **Write down the equation of element stiffness matrix as applied to 2D plane element.**

   The equation of element stiffness matrix for 2D plane element is

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

4. **Define degree of freedom of the structure with an example.**
   What is degree of kinematic indeterminacy and give an example.

   Degree of freedom is defined as the least no of independent displacements required todefine the deformed shape of a structure.
   There are two types of DOF: (a) Nodal type DOF and (b) Joint type DOF.

For example:

i = r – e where, r = no of reactions, e = no of equilibrium conditions r = 4 and e = 3
i = 4 – 3 = **1**

5. **Write a short note on global stiffness matrices.**

   The size of the global stiffness matrix (GSM) = No: of nodes x Degrees of freedom per node.

6. **Write a note on element stiffness matrix.**

$$K = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix}$$

The element stiffness is K_1, K_2, K_3 etc......

7. **List out the properties of rotation matrix.**
   - Matrix multiplication has no effect on the zero vectors (the coordinates of the origin).
   - It can be used to describe rotations about the origin of the coordinate system.
   - Rotation matrices provide an algebraic description of such rotations.
   - They are used extensively for computations.
   - Rotation matrices are square matrices with real entries.

8. **What are the basic unknowns in stiffness matrix method?**

   In the stiffness matrix method nodal displacements are treated as the basic unknowns forthe solution of indeterminate structures.

9. **Define stiffness coefficient 'kij'.**

   Stiffness coefficient 'kij' is defined as the force developed at joint 'i' due to unit displacement at joint 'j' while all other joints are fixed.

10. **What is the basic aim of the stiffness method?**

    The aim of the stiffness method is to evaluate the values of generalized coordinates 'r' knowing the structure stiffness matrix 'k' and nodal loads 'R' through the structure equilibrium equation.

    $$\{R\} = [K]\ \{r\}$$

11. **What is the displacement transformation matrix?**

    The connectivity matrix which relates the internal displacement 'q' and the external displacement 'r' is known as the displacement transformation matrix 'a'.

    $$\{q\} = [a]\ \{r\}$$

12. **How are the basic equations of stiffness matrix obtained?**

    The basic equations of stiffness matrix are obtained as:
    - Equilibrium forces
    - Compatibility of displacements
    - Force displacement relationships

13. **What is meant by generalized coordinates?**

    For specifying a configuration of a system, a certain minimum no of independent coordinates are necessary. The least no of independent coordinates that are needed to specify the configuration is known as generalized coordinates.

14. **Write about the force displacement relationship.**

    The relationship of each element must satisfy the stress-strain relationship of the elementmaterial.

15. **Compare flexibility method and stiffness method.**

    Flexibility matrix method:
    - The redundant forces are treated as basic unknowns.

- The number of equations involved is equal to the degree of static indeterminacy of thestructure.
- The method is the generalization of consistent deformation method.
- Different procedures are used for determinate and indeterminate structures

Stiffness matrix method:

- The joint displacements are treated as basic unknowns
- The number of displacements involved is equal to the no of degrees of freedom of the structure
- The method is the generalization of the slope deflection method.
- The same procedure is used for both determinate and indeterminate structures.

16. Is it possible to develop the flexibility matrix for an unstable structure?

In order to develop the flexibility matrix for a structure, it has to be stable and determinate.

17. What is the relation between flexibility and stiffness matrix?

The element stiffness matrix 'k' is the inverse of the element flexibility matrix 'f' and is givenby f = 1/k or k = 1/f.

18. List the properties of the stiffness matrix.

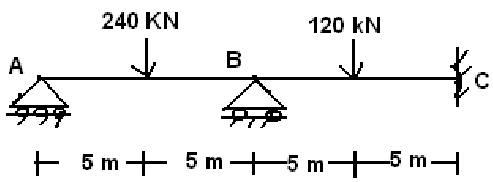
- The properties of the stiffness matrix are:
- It is a symmetric matrix
- The sum of elements in any column must be equal to zero.
- It is an unstable element therefore the determinant is equal to zero.

19. Why the stiffness matrix method is also called equilibrium method or displacement method?

Stiffness method is based on the superposition of displacements and hence is also knownas the displacement method. And since it leads to the equilibrium equations the method is also known as equilibrium method.

Part B – 16 Mark Questions and Answers

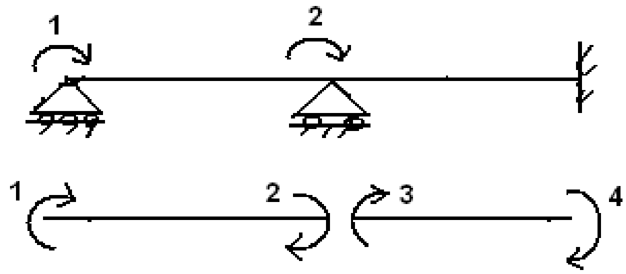
1. Analyse the continuous beam shown in figure using displacement method.



EI = constant

Solution:

Step1: Assign coordinates :



Step2: Fixed End Moment :

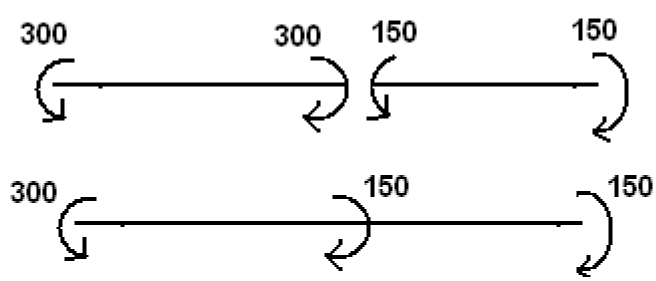
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{240 \text{ x } 10}{8} = -300 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{240 \text{ x } 10}{8} = 300 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{120 \text{ x } 10}{8} = -150 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell}{8} = \frac{120 \text{ x } 10}{8} = 150 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -300 \\ 150 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K) :

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}$$

Step 6 :System stiffness matrix

$$(J) = A^T K A$$

$$= EI \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0.4 & \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 0.4 & \\ & 0. \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 2.8 & -0.7 \\ -6 & 1 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0} \right]$$

$$= \frac{1}{EI}\begin{bmatrix} 2.86 & -0.71 \\ -0.71 & 1.43 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -300 \\ 150 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \\ 0 & 0.4 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

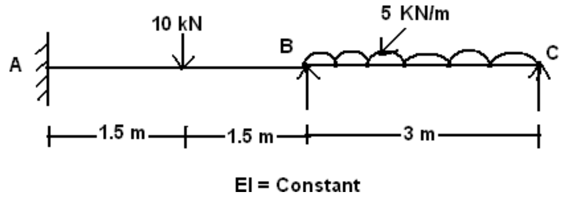
$$P = \begin{bmatrix} 300 \\ 21.9 \\ -171 \\ -85.5 \end{bmatrix}$$

Step 9 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -30 \\ 300 \\ -150 \\ 150 \end{bmatrix} + \begin{bmatrix} 30 \\ 21.9 \\ -171 \\ -85.5 \end{bmatrix}$$

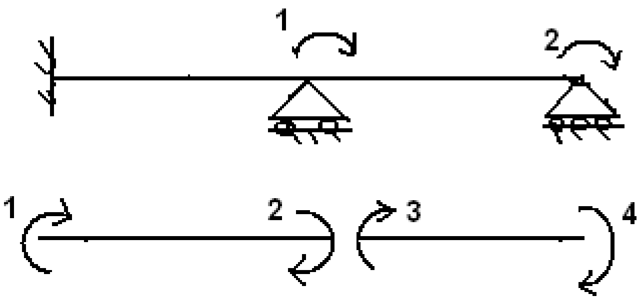
$$M = \begin{bmatrix} 0 \\ 321.9 \\ -321 \\ 64.5 \end{bmatrix}$$

2. Analyse the continuous beam ABC shown in figure by stiffness method and also draw the shear force diagram.



Solution:

Step1: Assign coordinates:



Step2: Fixed End Moment:

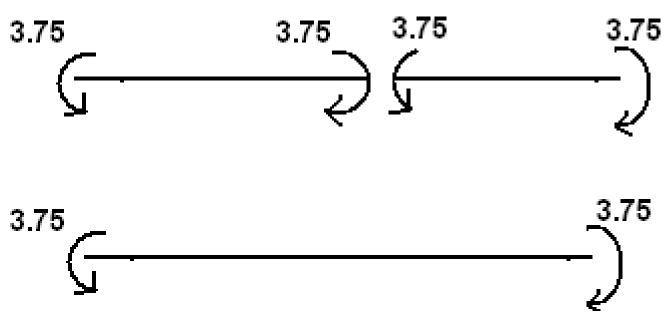
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{10 \times 3}{8} = -3.75 \ kNm$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{10 \times 3}{8} = 3.75 \ kNm$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{5 \times 3^2}{12} = -3.75 \ kNm$$

$$M_{FCB} = \frac{w\ell^2}{1} = \frac{5 \times}{3^2} = 3.75$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} 0 \\ 3.75 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step5: Stiffness matrix (K) :

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}$$

Step 6 : System stiffness matrix

$(J) = A^T K A$

$$= EI \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.67 & 1.33 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 2.66 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 0.43 & -0.21 \\ -1 & 7 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0} \right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.431 & -0.217 \\ -0.217 & 0.861 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 3.75 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

$$= \begin{bmatrix} 0.67 & 0 \\ 1.33 & 0 \\ 1.33 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

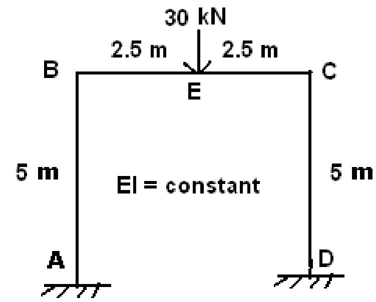
$$P = \begin{bmatrix} 0.545 \\ 1.082 \\ -1.081 \\ -3.75 \end{bmatrix}$$

Step 9 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -3.7 \\ 3.7 \\ -5 \\ 3.7 \end{bmatrix} + \begin{bmatrix} 0.545 \\ 1.082 \\ - \\ 1.081 \end{bmatrix}$$

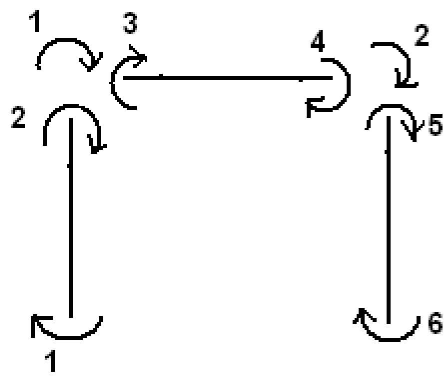
$$M = \begin{bmatrix} -3.205 \\ 4.832 \\ -4.832 \\ 0 \end{bmatrix}$$

3. Analyse the portal frame ABCD shown in figure by stiffness method and also draw the bending moment diagram.



Solution:

Step1: Assign coordinates :



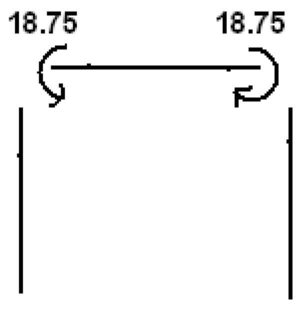
Step 2: Fixed End Moment :

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{30 \text{ x } 5}{8} = -18.75 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell}{8} = \frac{30 \text{ x } 5}{8} = 18.75 \text{ kNm}$$

$$M_{FAB} = M_{FBA} = \quad = M_{FD} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -18.75 \\ 18.75 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step5: Stiffness matrix (K) :

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}$$

Step 6 :System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.4 & 0.8 & 0.8 & 0.4 & 0 & 0 \\ & 0 & 0.4 & 0.8 & 0.8 & \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 1.6 & 0.4 \\ 0.4 & \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.6 & -0.17 \\ - & \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0} \right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.67 & -0.17 \\ -0.17 & 0.67 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -18.75 \\ 18.75 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

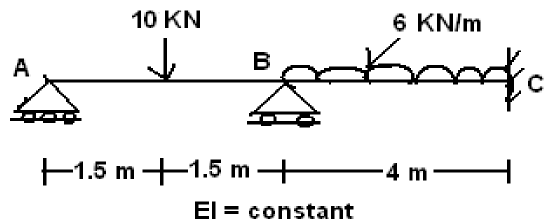
$$= \begin{bmatrix} 0.4 & 0 \\ 0.8 & 0 \\ 0.8 & 0.4 \\ 0.4 & 0.8 \\ 0 & 0.8 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

$$P = \begin{bmatrix} 6.3 \\ 12.6 \\ 6.3 \\ -6.3 \\ -12.6 \\ -6.3 \end{bmatrix}$$

Step 9: Final Moments (M):

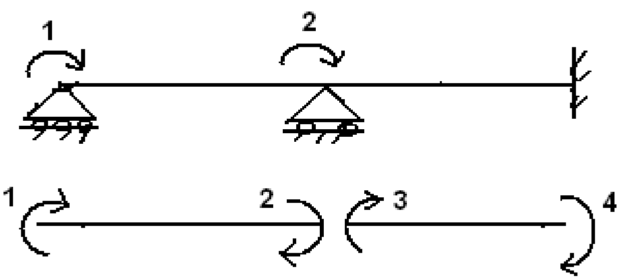
$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -18.75 \\ 18.75 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6.3 \\ 12.6 \\ 6.3 \\ -6.3 \\ -12.6 \\ -6.3 \end{bmatrix} = \begin{bmatrix} 6.3 \\ 12.6 \\ -12.5 \\ 12.5 \\ -12.6 \\ -6.3 \end{bmatrix}$$

4. Analyse the continuous beam ABC shown in figure by stiffness method and also sketch the bending moment diagram.



Solution:

Step1: Assign coordinates :



Step 2: Fixed End Moment :

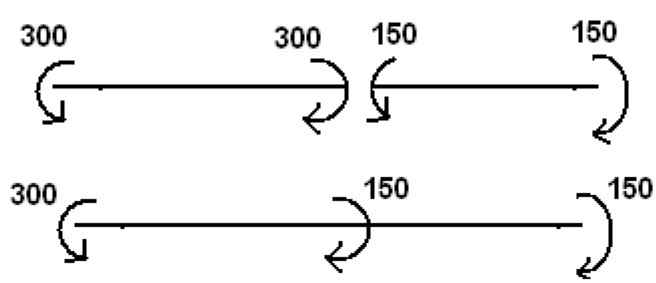
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{10 \times 3}{8} = -3.75 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{10 \times 3}{8} = 3.75 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{6 \times 4^2}{12} = -8 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell^2}{12} = \frac{6 \times 4^2}{12} = 8 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -3.75 \\ -4.25 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step5: Stiffness matrix (K) :

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}$$

Step 6 :System stiffness matrix

$(J) = A^T K A$

$$= EI\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.6 & 1.33 & 1 & 0. \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & \end{bmatrix}$$

$$J = EI\begin{bmatrix} 1.33 & 0.6 \\ 0.67 & \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.879 & 0.253 \\ -0.253 & 0.502 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.879 & -0.253 \\ -0.253 & 0.502 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -3.75 \\ -4.25 \end{Bmatrix}\right]$$

$$\frac{1}{EI}\begin{bmatrix} 2.22 \\ \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2.221 \\ 1.185 \end{bmatrix}$$

$$= \begin{bmatrix} 1.33 & 0.67 \\ 0.67 & 1.33 \\ 0 & 1 \\ 0 & 0. \end{bmatrix}\begin{bmatrix} 2.221 \\ 1.18 \end{bmatrix}$$

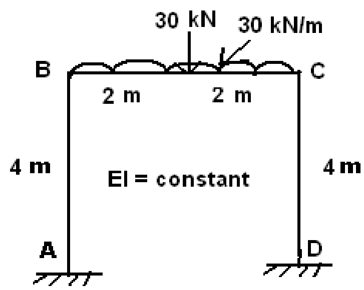
$$P = \begin{bmatrix} 3.75 \\ 3.06 \\ 1.185 \\ 0.59 \end{bmatrix}$$

Step 9 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -3.7 \\ 3.75 \\ -8 \\ 8 \end{bmatrix} + \begin{bmatrix} 3.7 \\ 3.06 \\ 1.185 \\ 0.59 \end{bmatrix}$$

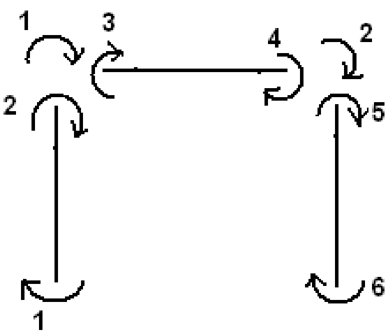
$$M = \begin{bmatrix} 0 \\ 6.81 \\ -6.81 \\ 8.59 \end{bmatrix}$$

5. **Analyse the portal frame ABCD shown in figure by stiffness method and also sketch the bending moment diagram.**



Solution:

Step1: Assign coordinates :



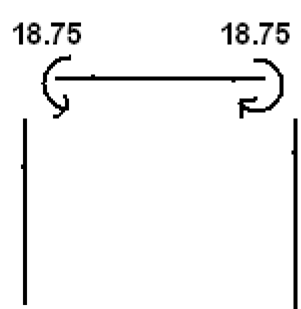
Step 2: Fixed End Moment :

$$M_{FBC} = -\left[\frac{w\ell}{8} + \frac{w\ell^2}{1}\right] = -\left[\frac{30 \text{ x } 4}{8} + \frac{30 \text{ x } 4^2}{12}\right] = -55$$

$$M_{FCB} = \left[\frac{w\ell}{8} + \frac{w\ell^2}{1}\right] = \left[\frac{30 \text{ x } 4}{8} + \frac{30 \text{ x } 4^2}{12}\right] = 55$$

$$M_{FAB} = M_{FBA} = \ 2 \ = M_{FD} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -55 \\ 55 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step5: Stiffness matrix (K) :

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

Step 6 :System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.5 & 1 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 1 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 2 & \\ 0.5 & 2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.5 & -0.13 \\ - & \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0} \right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.53 & -0.13 \\ -0.13 & 0.53 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -55 \\ 55 \end{Bmatrix}\right]$$

$$\left[\frac{1}{EI}\right]\begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \\ 1 & 0.5 \\ 0.5 & 1 \\ 0 & 1 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

$$P = \begin{bmatrix} 18.15 \\ 36.3 \\ 18.15 \\ -18.15 \\ -36.3 \\ -18.15 \end{bmatrix}$$

Step 9: Final Moments (M):

$$= \mu + M_P = \begin{bmatrix} 0 \\ 0 \\ -55 \\ 55 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 18.15 \\ 36.3 \\ 18.15 \\ -18.15 \\ -36.3 \\ -18.15 \end{bmatrix} = \begin{bmatrix} 18.15 \\ 36.3 \\ -36.3 \\ 36.45 \\ -36.3 \\ -18.15 \end{bmatrix}$$

UNIT 3,4 - ARCHES, CABLE AND SUSPENSION BRIDGE

Part A- 2 Mark Questions and Answers

1. **Give any two examples of beams curved in plan.**
    1. Beams in a bridge negotiating a curve
    2. Ring beams supporting a water tank
    3. Beams supporting corner lintels
    4. Beams in ramps

2. **What is the nature of forces in the cables?**

    Cables of cable structures have only tension and no compression or bending.

3. **Define tension coefficient. For what type of structures tension coefficient method is employed?**

    The tension coefficient for a member of a truss is defined as the pull or tension in themember divided by its length, i. e. the force in the member per unit length.

4. **What are the components of forces acting on the beams curved in plan and show the sign conventions of these forces?**
    1. Bending moments
    2. Shear forces
    3. Torsional moments

5. **Define a space frame and what is the nature of joint provided in the space trusses?**

    A space frame is a structure built up of hinged bars in space. It is three dimensional generalization of a truss.

    Socket joint is provided in the space trusses.

6. **What are the types of stiffening girders?**
    1. Suspension bridges with three hinged stiffening girders
    2. Suspension bridges with two hinged stiffening girders

7. **What are the methods available for the analysis of space trusses?**

    Tension co-efficient method is available for the analysis of space trusses.

8. **What is the need for cable structures?**
    1. The main load bearing member.
    2. Flexible throughout.
    3. It can take only direct tension and cannot take any bending moment.

9. What are cable structures?

Long span structures subjected to tension and uses suspension cables for supports. Examples ofcable structures are suspension bridges, cable stayed roof.

10. What is the true shape of cable structures?

Cable structures especially the cable of a suspension bridge is in the form of a catenary. Catenaryis the shape assumed by a string / cable freely suspended between two points.

11. Mention the different types of cable structures.

(a) Cable over a guide pulley
(b) Cable over a saddle

12. Briefly explain cable over a guide pulley.

1. Tension in the suspension cable = Tension in the anchor cable
2. The supporting tower will be subjected to vertical pressure and bending due to net horizontal cable tension.

13. Briefly explain cable over saddle.

1. Horizontal component of tension in the suspension cable = Horizontal component of tension in the anchor cable
2. The supporting tower will be subjected to only vertical pressure due to cable tension.

14. What are the main functions of stiffening girders in suspension bridges?

1. They help in keeping the cables in shape
2. They resist part of shear force and bending moment due to live loads.

15. Differentiate between plane truss and space truss.

Plane truss

1. All members lie in one plane
2. All joints are assumed to be hinged.

Space truss

- This is a three dimensional truss
- All joints are assumed to be ball and socketed.

16. What are the significant features of circular beams on equally spaced supports?

1. Slope on either side of any support will be zero.
2. Torsional moment on every support will be zero

17. Give the expression for calculating equivalent UDL on a girder.

The tension developed in the cable is given by

$$T = \sqrt{H^2 + V^2}$$

Where, H = horizontal component and V = vertical component.

18. Define tension co-efficient.

The tension co-efficient for a member of a truss is defined as the pull or tension in that member divided by its length.

19. What are cables made of?

Cables can be of mild steel, high strength steel, stainless steel, or polyester fibres. Structural cables are made of a series of small strands twisted or bound together to form a much larger cable.

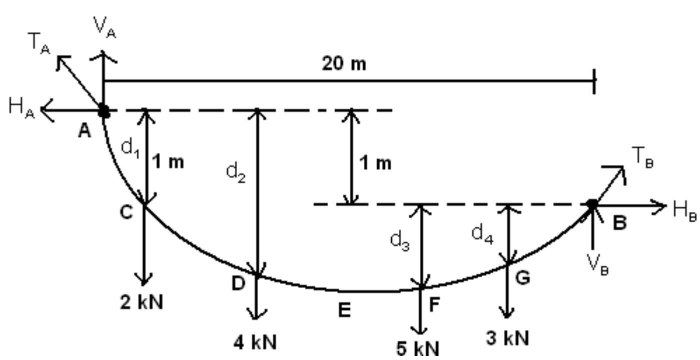
Steel cables are either spiral strand, where circular rods are twisted together or locked coil strand, where individual interlocking steel strands form the cable (often with a spiral strand core).

20. Give the types of significant cable structures

1. Suspension bridges
2. Cable-stayed beams or trusses
3. Cable trusses
4. Draped cables
5. Straight tensioned cables

PART B - 8 Mark Questions and Answers

1. A suspension cable is supported at two point "A" and "B", "A" being one metre above "B". the distance AB being 20 m. the cable is subjected to 4 loads of 2 kN, 4 kN, 5 kN and 3 kN at distances of 4 m, 8 m, 12 m and 16 m respectively from "A". Find the maximum tension in the cable, if the dip of the cable at point of application of first loads is 1 m with respect to level at A. find also the length of the cable



Solution

Reactions

$$\sum V = 0$$

$$V_A + V_B = 14$$

$$\sum M @ B = 0$$

$$(V_A \times 20) - (H \times 1) - (2 \times 16) - (4 \times 12) - (5 \times 8) - (3 \times 4) = 0$$

$$H = 33 \text{ kN}$$

$$V_A = 8.25 \text{ kN}$$

$$V_B = 5.75 \text{ kN}$$

Maximum Tension in the cable

$$= \sqrt{V_A^2 + H^2} = \sqrt{8.25^2 + 33^2} = 34.02 \text{ kN}$$

$$= \sqrt{V_B^2 + H^2} = \sqrt{5.75^2 + 33^2} = 33.49 \text{ kN}$$

$$\text{Tension in the cable, } T_{max} = 34.09 \text{ kN.}$$

Length of the cable

Here, $d_1 = 1m$

Equating moments about D to zero,

$(8.25 \text{ x } 8) - (33 \text{ x } d_2) = 0$

$d_2 = 2\,m$

Equating moments about D to zero,

$d_3 = 1.39\,m$

Equating moments about D to zero,

$-5.75 \text{ x } 4) + (33 \text{ x } d_4) = 0$

$d_4 = 0.69\,m$

$AC = \sqrt{4^2 + 1^2} = 4.12 \text{ m}$

$CD = \sqrt{4^2 + 2^2} = 4.47 \text{ m}$

$FG = \sqrt{4^2 + 1.39^2} = 4.23 \text{ m}$

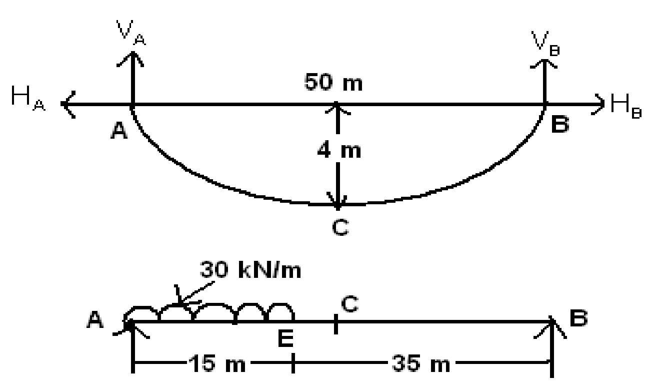
$GB = \sqrt{4^2 + 0.69^2} = 4.06 \text{ m}$

Length of the cable, L $= AC + CD + FG + BG + DF$

$= 4.12 + 4.47 + 4.23 + 4.06 + 4$

$= 20.88\,m$

2. **A suspension bridge has a span 50 m with a 15 m wide runway. It is subjected to a load of 30kN/m including self-weight. The bridge is supported by a pair of cables having a centraldip of 4 m. find the cross sectional area of the cable necessary if the maximum permissiblestress in the cable materials is not to exceed 600 MPa.**



Solution

Reactions

$\sum V = 0$

$V_A + V_B = 450$

$\sum M @ A = 0$

$V_B = 67.5$ kN

$V_A = 382.5$ kN

$\sum H = 0$

$H_A = H_B$

$H = 421.87$ kN

Maximum tension in cable

$T_A = 569.46$ kN

$T_B = 427.24$ kN

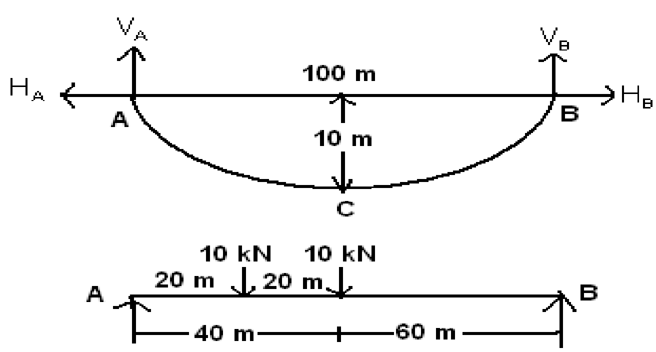
Maximum Tension in cable = 569.46 kN

$T_{max} = \sigma \, . \, A$

$= 596460/600$

$= 949.1 \ mm^2.$

3. **A three hinged stiffening girder of a suspension bridge of 100 m span subjected to two point loads 10 kN each placed at 20 m and 40 m respectively from the left hand hinge. Determine the bending moment and shear force in the girder at section 30 m from each end. Also determine the maximum tension in the cable which has a central dip of 10 m.**



Solution

$\sum V = 0$

$V_A + V_B = 20$

$\sum M @ B = 0$

$(V_A \times 100) - (10 \times 80) - (10 \times 60) = 0$

$V_A = 14$ kN

$V_B = 6$ kN

$\sum H = 0$

$H_A = H_B$

$\sum M @ C = 0$

$(V_A \times 50) - (H \times 10) - (10 \times 30) - (10 \times 10) = 0$

$H = 30$ kN

Shear Force at 30m from the left hinge

$V_{30} = V_A - 10 - H \tan \theta$

$\tan \theta = 0.16$

$V_{30} = 14 - 10 - (30 \times 0.16) = -0.8$ kN

Shear Force at 30m from the right hinge

$V_{30} = V_B - H \tan \theta$

$V_{30} = 6 - (30 \times 0.16) = 1.2$ kN

Bending Moment at 30m from the left hinge

$BM_{30} = V_A \times 30 - H \times y - 10 \times 10$

At $X = 30$ m and $y = 8.4$m

$BM_{30} = (14 \times 30) - (30 \times 8.4) - 100 = 68$ kNm

Bending Moment at 30m from the left hinge

$BM_{30} = -V_B \times 30 - H \times y$

$BM_{30} = -(6 \times 30) - (30 \times 8.4) = 72$ kNm

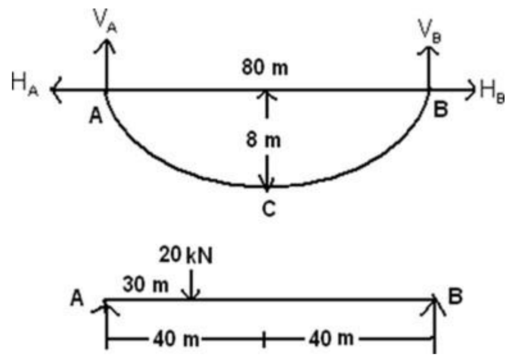
Maximum Tension in the cable

$$T_A = \sqrt{V_A^2 + H^2} = \sqrt{14^2 + 30^2} = 33.11 \text{ kN}$$

$$T_B = \sqrt{V_B^2 + H^2} = \sqrt{6^2 + 30^2} = 30.59 \text{ kN}$$

Tension in the cable, $T_{max} = 33.11$ kN.

4. **A suspension bridge cable of span 80 m and central dip 8 m is suspended from the same level at two towers. The bridge cable is stiffened by a three hinged stiffening girder which carries a single concentrated load of 20 kN at a point of 30 m from one end. Sketch the SFD for the girder.**



Solution

Reactions

$\sum V = 0$

$V_A + V_B = 20$

$\sum M @ B = 0$

$(V_A \times 80) - (20 \times 50) = 0$

$V_A = 12.5$ kN

$V_B = 7.5$ kN

$\sum H = 0$

$H_A = H_B$

$\sum M @ C = 0$

$(V_A \times 40) - (H \times 8) - (20 \times 10) = 0$

$H = 37.5$ kN

Shear Force at 40m from the left hing

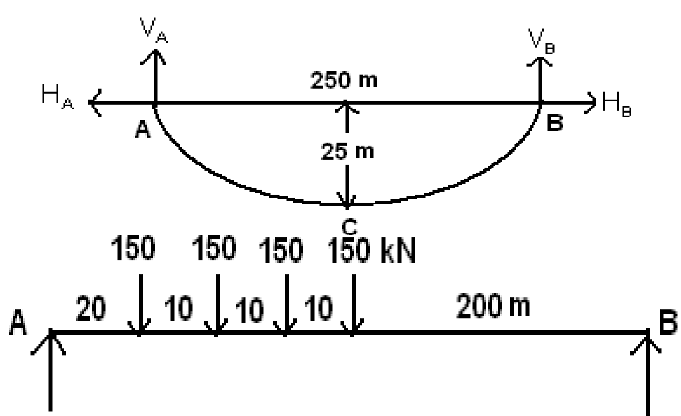
$V_{40} = V_A - 20 - H \tan\theta$

$\tan\theta = 0$

$V_{40} = 12.5 - 20 - (37.5 \times 0)$

$V_{40} = -7.5$ kN

5. A suspension bridge 0f 250 m span has two nos. of three hinged stiffening girders supported by cables with a central dip of 25 m. if 4 point loads of 300 kN each are placed at the centre line of the roadway at 20, 30, 40 and 50 m from left hand hinge. Find the shear force and bending moment in each girder at 62.5 m from each end. Calculate also the maximum tension in the cable.



Solution

Reactions

$\sum V = 0$

$V_A + V_B = 600$

$\sum M @ B = 0$

$(V_A \ x\ 250) - (150\ x\ 230) - (150\ x\ 220) - (150\ x\ 210) - (150\ x\ 200) = 0$

$V_A = 516$ kN

$V_B = 84$ kN

$\sum H = 0$

$H_A = H_B$

$\sum M @ C = 0$

$(V_A\ x\ 125) - (H\ x\ 25) - (150\ x\ 105) - (150\ x\ 95) - (150\ x\ 85) - (150\ x\ 75) = 0$

$H = 420$ kN

Shear force at 62.5m from left hinge

$V_{62.5} = V_A - 150 - 150 - 150 - 150 - H \tan\theta$

$\tan\theta = 0.2$

$V_{62.5} = 516 - 150 - 150 - 150 - 150 - (420\ x\ 0.2)$

$V_{62.5} = -168$ kN

Shear force at 62.5m from right hinge

$V_{62.5} = V_B - H \tan\theta$

$V_{62.5} = 84 - (420\ x\ 0.2)$

$V_{62.5} = 0$

Bending at 62.5m from left hinge

$BM_{62.5} = V_A\ x\ 62.5 - (150\ x\ 42.5) - (150\ x\ 32.5) - (150\ x\ 22.5) - (150\ x\ 12.5) - H\ x\ y$

At x = 62.5m and y = 18.75m

$BM_{62.5} = 516\ x\ 62.5 - (150\ x\ 42.5) - (150\ x\ 32.5) - (150\ x\ 22.5) - (150\ x\ 12.5) - (420\ x\ 18.75)$

$BM_{62.5} = 7875$ kNm

Bending at 62.5m from right hinge

$BM_{62.5} = - V_B\ x\ 62.5 + H\ x\ y$

$BM_{62.5} = -(84\ x\ 62.5) + (420\ x\ 18.75)$

$BM_{62.5} = 2625$ kNm

Maximum tension in the cable

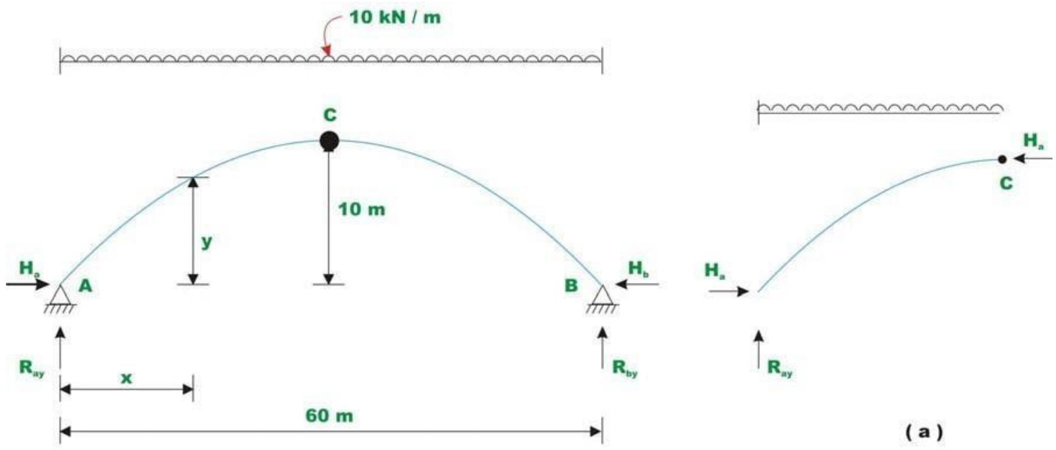
$Hd = Wl^2/8$

$W = H\ x\ d\ x\ 8/l^2 = 420\ x\ 25\ x\ 8/\ 250^2 = 1.344$ kN/m

$V_A = V_B = Wl/2 = 1.344 \times 250 / 2 = 168$ kN

$$T_{max} = \sqrt{V_A^2 + H^2} = \sqrt{168^2 + 420^2} = 452.32 \text{ kN}$$

6. **A three-hinged parabolic arch of uniform cross section has a span of 60 m and a rise of 10 m. It is subjected to uniformly distributed load of intensity 10 kN/m. Show that the bending moment is zero at any cross section of the arch.**



Solution
Reactions

Taking moment of all the forces about hinge A, yields

$$R_{ay} = R_{by} = \frac{10 \times 60}{2} = 300 \text{ kN}$$

Taking moment of forces left of hinge C about C, one gets

$$R_{ay} \times 30 - H_a \times 10 - 10 \times 30 \times \frac{30}{2} = 0$$

$$H_a = \frac{300 \times 30 - 10 \times 30 \times \left(\frac{30}{2}\right)}{10}$$

$$= 450 \text{ kN}$$

From $\sum Fx = 0$ one could write, $H_b = 450$ kN .

The shear force at the mid span is zero.

Bending moment

The bending moment at any section x from the left end is,

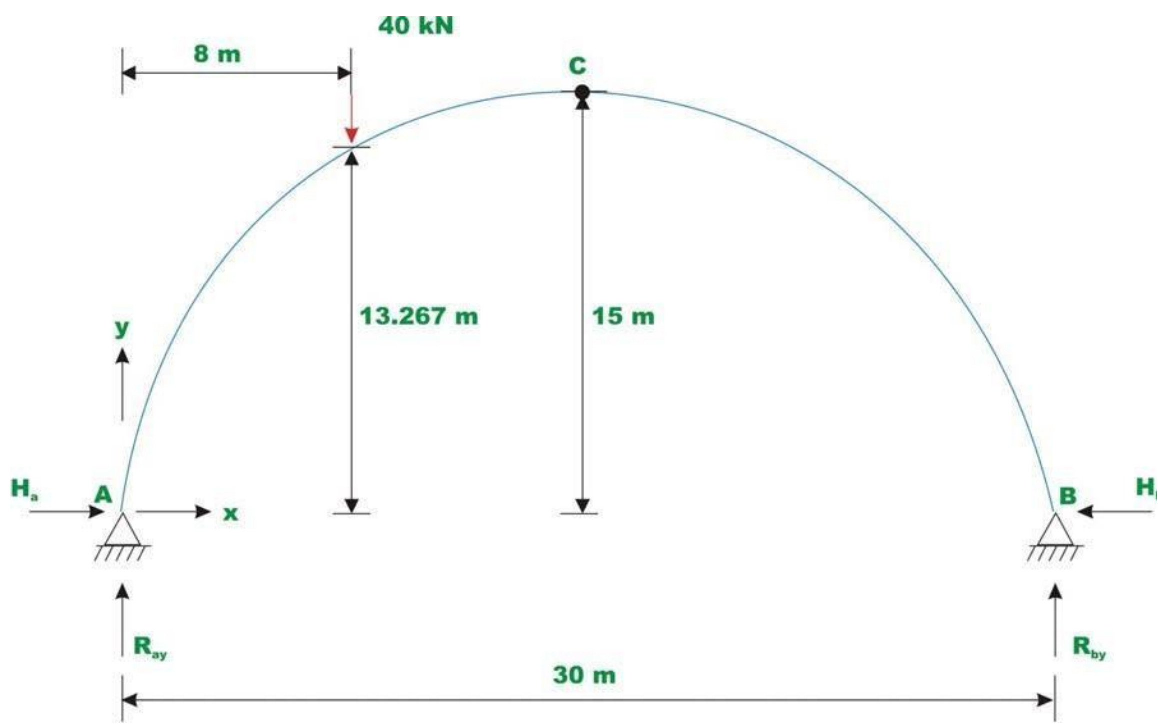
The equation of the three-hinged parabolic arch is

$$y = \frac{2}{3}x - \frac{10}{30^2}x^2$$

$$M_x = 300x - \left(\frac{2}{3}x - \frac{10}{30^2}x^2\right)450 - 5x^2$$

$$= 300x - 300x + 5x^2 - 5x^2 = 0$$

7. **A three-hinged semicircular arch of uniform cross section is loaded as shown in Figure. Calculate the location and magnitude of maximum bending moment in the arch.**



Reactions

Taking moment of all the forces about hinge B leads to,

$$R_{ay} = \frac{40 \times 22}{30} = 29.33 \text{ kN } (\uparrow)$$

$$\sum Fy = 0 \qquad \Rightarrow R_{by} = 10.67 \text{ kN } (\uparrow)$$

Bending moment

Now making use of the condition that the moment at hinge C of all the forces left of hinge C is zero gives,

$$M_c = R_{ay} \times 15 - H_a \times 15 - 40 \times 7 = 0$$

$$H_a = \frac{29.33 \times 15 - 40 \times 7}{15} = 10.66 \text{ kN } (\rightarrow)$$

Considering the horizontal equilibrium of the arch gives,

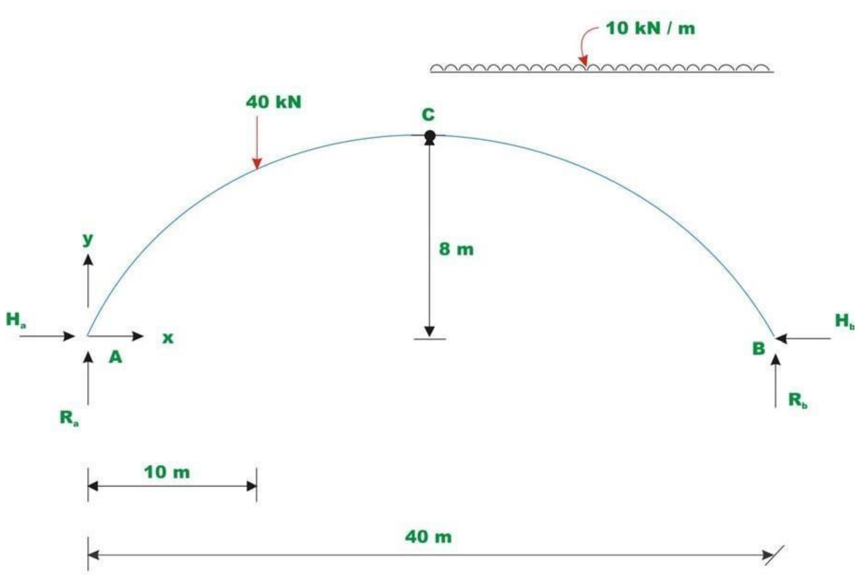
$$H_b = 10.66 \text{ kN } (\leftarrow)$$

The maximum positive bending moment occurs below D and it can be calculated by taking moment of all forces left of D about D.

$$M_D = R_{ay} \times 8 - H_a \times 13.267$$

$$= 29.33 \times 8 - 10.66 \times 13.267 = 93.213 \text{ kN}$$

8. **A three-hinged parabolic arch is loaded as shown in Figure. Calculate the location and magnitude of maximum bending moment in the arch. Draw bending moment diagram.**



Solution

Reactions

Taking A as the origin, the equation of the three-hinged parabolic arch is givenby,

$$y = \frac{8}{10}x - \frac{8}{400}x^2$$

Taking moment of all the forces about hinge B leads to,

$$R_{ay} = \frac{40 \times 30 + 10 \times 20 \times \left(20/2\right)}{40} = 80 \text{ kN} \quad (\uparrow)$$

$$\sum Fy = 0 \Rightarrow R_{by} = 160 \text{ kN} \quad (\uparrow)$$

Now making use of the condition that, the moment at hinge C of all the forcesleft of hinge C is zero gives,

$$M_c = R_{ay} \times 20 - H_a \times 8 - 40 \times 10 = 0$$

$$H_a = \frac{80 \times 20 - 40 \times 10}{8} = 150 \quad \text{kN} \quad (\rightarrow)$$

Considering the horizontal equilibrium of the arch gives,

$$H_b = 150 \text{ kN} \quad (\leftarrow)$$

Location of maximum bending moment

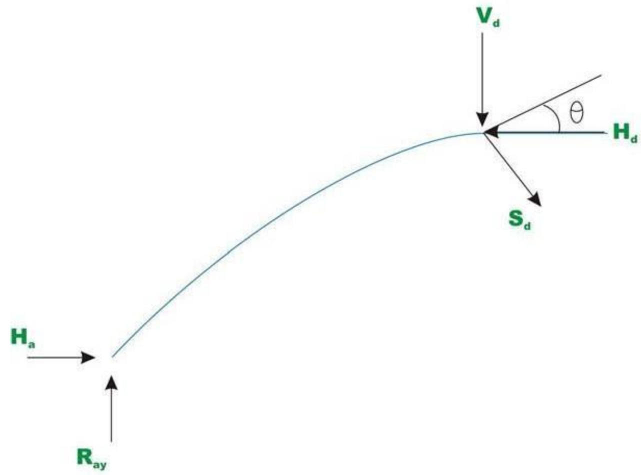
According to calculus, the necessary condition for extreme (maximum or minimum) is that $\frac{\partial M_x}{\partial x} = 0$.

$$= 40 - 4x = 0$$

$$x = 10 \text{ m.}$$

$$M_{max} = 200 \text{kNm}$$

Shear force at D just left of 40 kN load



The slope of the arch at D is evaluated by,

$$\tan\theta = \frac{dy}{dx} = \frac{8}{10} - \frac{16}{400}x$$

Substituting $x = 10$ m. in the above equation, $\theta_D = 21.8^0$

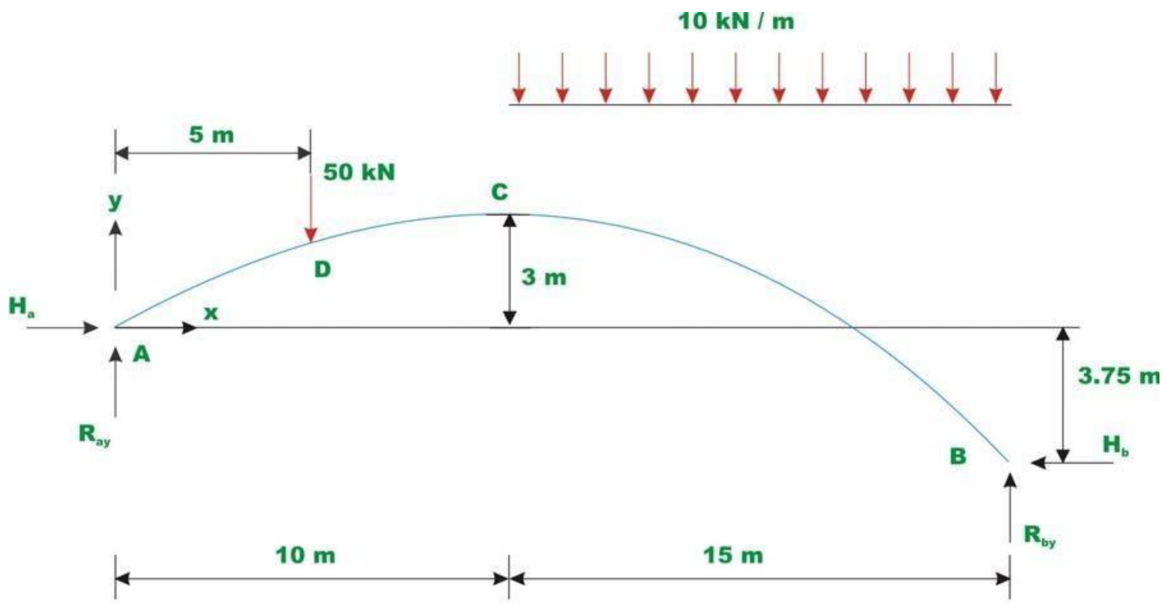
Shear force S_d ***at left of*** D **is**

$$S_d = H_a \sin\theta - R_{ay}\cos\theta$$

$$S_d = 150\sin(21.80) - 80\cos(21.80)$$

$$= -18.57 \text{ kN.}$$

9. **A three-hinged parabolic arch of constant cross section is subjected to a uniformly distributed load over a part of its span and a concentrated load of 50 kN, as shown in Figure. The dimensions of the arch are shown in the figure. Evaluate the horizontal thrust and the maximum bending moment in the arch.**



Solution

Reactions

Taking A as the origin, the equation of the parabolic arch may be written as,

$$y = -0.03x^2 + 0.6x$$

Taking moment of all the loads about B leads to,

$$R_{ay} = \frac{1}{25}\left[50\times 20 + 10\times 15\times \frac{15}{2} - H_a \times 3.75\right]$$

$$= \frac{1}{25}\left[2125 - 3.75H_a\right]$$

Taking moment of all the forces right of hinge C about the hinge C and setting $M_c = 0$ leads to,

$$R_{by}\times 15 - 6.75H_b - 10\times 15\times 15/2 = 0$$

$$R_{by} = \frac{1}{15}\left[1125 + 6.75H_b\right]$$

Since there are no horizontal loads acting on the arch,

$$H_a = H_b = H$$

Applying $\sum Fy = 0$ for the whole arch,

$$R_{ay} + R_{by} = 10 \times 15 + 50 = 200$$

$$1/25\left[2125 - 3.75\,H\right] + 1/15\left[1125 + 6.75\,H\right] = 200$$

$$85 - 0.15\,H + 75 + 0.45\,H = 200$$

$$H = \frac{40}{0.3} = 133.33\ \text{kN}$$

$$R_{ay} = 65.0\ \text{kN}$$

$$R_{by} = 135.0\ \text{kN}$$

Bending moment

Span AD

Bending moment at any cross section in the span AD is

$$M = R_{ay}\,x - H_a\,(-0.03x^2 + 0.6\,x) \qquad 0 \le x \le 5$$

For, the maximum negative bending moment in this region,

$$R_{ay} - H_a\,(-0.06x + 0.6) = 0$$

$$x = 1.8748\ \text{m}$$

$$M = -14.06\ \text{kN.m.}$$

For the maximum positive bending moment in this region occurs at D,

$$M_D = R_{ay}\,5 - H_a\,(-0.03 \times 25 + 0.6 \times 5)$$

$$= +25.0\ \text{kN.m}$$

UNIT 5- INTRODUCTION TO FINITE ELEMENT METHOD

Part A- 2 Mark Questions and Answers

1. What is meant by Finite element method?

Finite element method (FEM) is a numerical technique for solving boundary value problems in which a large domain is divided into smaller pieces or elements. The solution is determined by assuming certain polynomials. The small pieces are called finite element and the polynomials are called shape functions.

2. List out the advantages of FEM.

- Since the properties of each element are evaluated separately different material properties can be incorporated for each element.
- There is no restriction in the shape of the medium.
- Any type of boundary condition can be adopted.

3. List out the disadvantages of FEM.

- The computational cost is high.
- The solution is approximate and several checks are required.

4. Mention the various coordinates in FEM.

- Local or element coordinates
- Natural coordinates
- Simple natural coordinates
- Area coordinates or Triangular coordinates
- Generalized coordinates

5. What are the basic steps in FEM?

- Discretization of the structure
- Selection of suitable displacement function
- Finding the element properties
- Assembling the element properties
- Applying the boundary conditions
- Solving the system of equations
- Computing additional results

6. What is meant by discretization?

Discretization is the process of subdividing the given body into a number of elements which results in a system of equivalent finite elements.

7. What are the factors governing the selection of finite elements?

- The geometry of the body
- The number of independent space coordinates
- The nature of stress variation expected

8. Define displacement function.

Displacement function is defined as simple functions which are assumed to approximate the displacements for each element. They may assume in the form of polynomials, or trigonometrical functions.

9. What are different types of elements used in FEM?

- One dimensional elements (1D elements)
- Two dimensional elements (2D elements)
- Three dimensional elements (3D elements)

10. Define Shape function.

Shape function is also called an approximate function or an interpolation function whose value is equal to unity at the node considered and zeros at all other nodes. Shape function is represented by Ni where i = node no.

11. What are the properties of shape functions?

The properties of shape functions are:

- The no of shape functions will be equal to the no of nodes present in the element.
- Shape function will have a unit value at the node considered and zero valueat other nodes.
- The sum of all the shape function is equal to 1.

12. Define aspect ratio.

Element aspect ratio is defined as the ratio of the largest dimension of the element to its smallest dimension.

13. What are possible locations for nodes?

- Point of application of concentrated load.
- Location where there is a change in intensity of loads
- Locations where there are discontinuities in the geometry of the structure

14. What are the characteristics of displacement functions?

- The displacement field should be continuous.
- The displacement function should be compatible between adjacent elements
- The displacement field must represent constant strain states of elements
- The displacement function must represent rigid body displacements of an element.

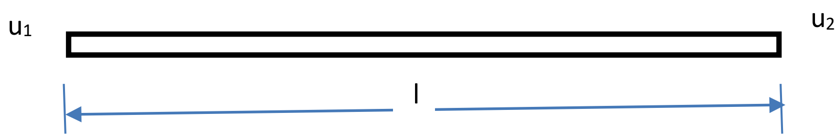
15. What is meant by plane strain condition?

Plane strain is a state of strain in which normal strain and shear strain directedperpendicular to the plane of body is assumed to be zero.

Part B- 16 Mark Questions and Answers

1. Derive the shape function and stiffness matrix for a 1 dimensional bar element

Consider a bar element with nodes 1 and 2 as shown with displacements of u_1 and u_2 at therespective nodes



The displacement u can be given as $u=a_0+a_1x$ coordinates.

$u = [1 \;\; x] \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$

At node 1, $u=u_1$, $x=0$

At node 2, $u=u_2$, $x=l$

$u_1=a_0$ and $u_2=a_0+a_1l$

In matrix form, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Substituting in eqn 2,

$$u = [1 \;\; x] \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$u = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix}$$

$u= [N_1 \; N_2] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ where $N_1= l-x/l$ and $N_2= x/l$

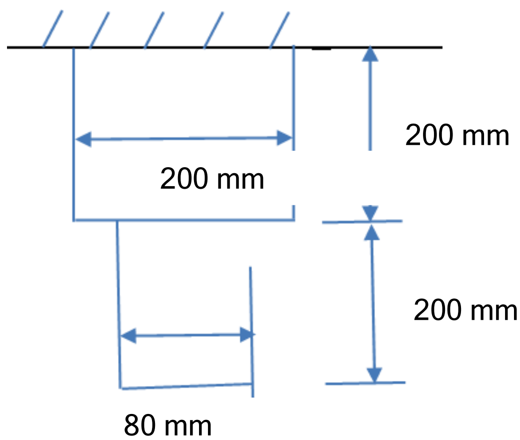
N_1 and $N_2$are the shape functions

$[B] = \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx}\right] = \left[\frac{-1}{l} \quad \frac{1}{l}\right]$ and $[B]^T = \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix}$

In one dimensional problems, [D] = Youngs Modulus and dv= Adx

$[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ which is the stiffness matrix for a one dimensional bar element.

2. **A thin steel plate of uniform thickness 25mm is subjected to a point load of 420N at mid depth as shown. The plate is also subjected to self weight. If Young's modulus, E=2 x 10^5N/mm^2 and unit weight density 0.8 x 10^{-4}N/mm^2. Calculate (i) Displacement at each nodal point (ii) Stresses in each element.**



Thickness t=25mm, A_1= 100 x 25=2500mm^2, A_2= 80x 25=2000mm^2 Point

load p= 420N

Young's modulus E=2 x 10^5 N/mm^2, Unit weight density= 0.8 x 10^{-4} N/mm^3

Body Force vector $\{F\} = \frac{\rho Al}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \frac{\rho_1 A_1 l_1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 36 \end{Bmatrix}$$

$$F_3 \qquad 16$$

A point load of 420N is acting at middepth at nodal point 2. Hence

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 456 \\ 16 \end{Bmatrix}$$

Apply boundary conditions at node 1 displacement is 0 and substituting the values off orces we get,

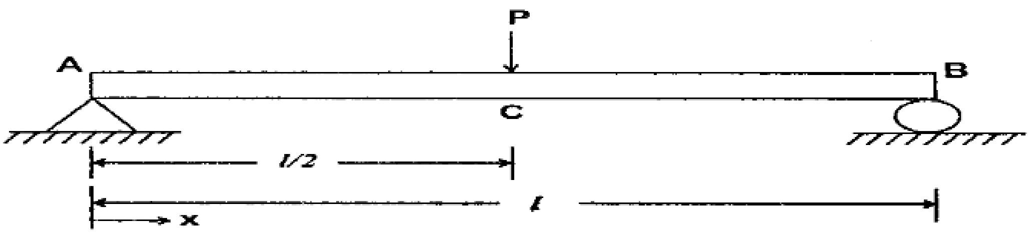
u_2= 1.888 x 10^{-4}mm , u_3= 1.9688 x 10^{-4}mm

Stress $\sigma = E\frac{du}{dx}$

$$\sigma_1 = \frac{2 \times 10^5 \times 1.888 \times 10^{-4}}{200} = 0.188 N/mm^2$$

$$\sigma_2 = \frac{2 \times 10^5 \times 1.968 \times 10^{-4} - 1.888 \times 10^{-4}}{200} = 0.008 N/mm^2$$

3. Find the deflection at the center of a simply supported beam of span length l, subjected to aconcentrated load P at its mid-point.



Solution

The total potential energy for a beam is given by, п = U - W

Strain energy for a beam, $U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$

Where E is the modulus of elasticity, I is the area moment of inertia of the beam section and y isthe deflection which can be expressed as,

$$y = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \dots.$$

to simplify the problem, consider the first three terms such as,

$$y = a_1 + a_2 x + a_3 x^2$$

The boundary conditions are $y = 0$ at $x = 0$ and $x = l$

$0 = a_1$ and $0 = a_2 l + a_3 l^2$ which gives $a_2 = - a_3 l$

Then y can be expressed as,

$y = - a_3 lx + a_3 x^2 = a_3 (x^2 - lx)$

Differentiating two times we get,

$\frac{dy}{dx} = a_3(2x - l) and \frac{d^2y}{dx^2} = 2a_3$

Then strain energy is given by,

$U = \frac{EI}{2}\int_0^l (2a_3)^2 dx = \frac{EI}{2} 4{a_3}^2 l = 2EI{a_3}^2 l$

Work done, $W = P * y_{\text{ at x=l/2}}$

$= P\, a_3(x^2 - lx)_{\text{ at x=l/2}}$

$$= Pa_3\left(\frac{l^2}{4} - \frac{l*l}{2}\right) \quad = -\,Pa_3\frac{l^2}{4}$$

The total potential energy is given by, π = U - W

$$= 2EIa_3{}^2l - \left(-\,Pa_3\frac{l^2}{4}\right) = 2EIa_3{}^2l + \,Pa_3\frac{l^2}{4}$$

For minimum potential energy condition,

$$\frac{\partial \pi}{\partial a_3} = 0$$

$$4EIa_3l = -P\frac{l^2}{4}$$

$$a_3 = -P\frac{l^2}{4} * \frac{1}{4EIl}$$

$$a_3 = -\frac{Pl}{16EI}$$

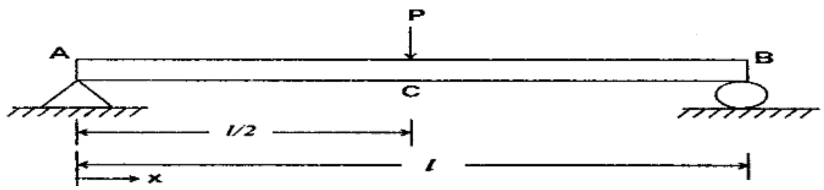
$$y = a_3(x^2 - lx) \quad = -\frac{Pl}{16EI}(x^2 - lx)$$

Maximum deflection occurs at x = l / 2

$$y_{max} = -\frac{Pl}{16EI}\left(\frac{l^2}{4} - l\frac{l}{2}\right) \quad = -\frac{Pl}{16EI}\left(-\frac{l^2}{4}\right)$$

$$y_{max} = -\frac{Pl^3}{64EI}$$

4. Find the deflection at the center of a simply supported beam of span length l subjected to aconcentrated load P at its mid-point using trail function from trigonometric series.



$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} + a_3 \sin\frac{5\pi x}{l} + \dots$

Potential energy as $\Pi = U - W$

Strain energy for a beam, $U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$

Differentiating the displacement function two times we get,

$$\frac{dy}{dx} = a\cos\frac{\pi x}{l} * \frac{\pi}{l} = \frac{a\pi}{l}\cos\frac{\pi x}{l}$$

$$\frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}$$

Then strain energy, $U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$

$$= \frac{EI}{2}\int_0^l \left(-a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}\right)^2 dx \qquad (2)$$

$$= \frac{EI}{2}\left(-a\frac{\pi^2}{l^2}\right)^2 \int_0^l \sin^2\frac{\pi x}{l}\,dx$$

Now, $\int_0^l \sin^2\frac{\pi x}{l}\,dx = \int_0^l \frac{1}{2}\left(1-\cos\frac{2\pi x}{l}\right)dx$ (Since $\sin^2 A = \frac{1-\cos 2A}{2}$)

$$= \frac{1}{2}\left[x - \left\{\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right\}\right]_0^l \quad = \frac{1}{2}\left[(l-0) - \frac{l}{2\pi}\{\sin 2\pi - \sin 0\}\right]$$

Strain energy, $U = \frac{EI}{2}\int_0^l \left(-a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}\right)^2 dx \quad = \frac{EI}{2}\left(-a\frac{\pi^2}{l^2}\right)^2 \frac{l}{2} \quad = \frac{a^2\pi^4 EI}{4l^3}$

Work Done, W = P * y_{max}

= P * $y_{\text{at } x=l/2}$ $= P\left(a\sin\frac{\pi x}{l}\right)_{at\ x=l/2}$ From Equation (1)

$$= Pa\sin\frac{\pi}{2} = Pa \quad (\text{Since } \sin\frac{\pi}{2} = 1)$$

The total potential energy, π = U - W

$$\pi = \frac{a^2\pi^4 EI}{4l^3} - Pa$$

$\frac{\partial \pi}{\partial a} = 0 \rightarrow \frac{2a\pi^4 EI}{4l^3} = P$ Therefore, $a = \frac{2pl^3}{\pi^4 EI}$

Maximum deflection occurs at x=l/2

Hence $y_{max} = \left(a\sin\frac{\pi x}{l}\right)_{at\ x=l/2}$ $y_{max} = \left(\frac{2pl^3}{\pi^4 EI}\sin\frac{\pi x}{l}\right)_{at\ x=l/2} \quad = \frac{2pl^3}{\pi^4 EI}\sin\frac{\pi}{2}$

Therefore, $y_{max} = \frac{pl^3}{48.7EI}$