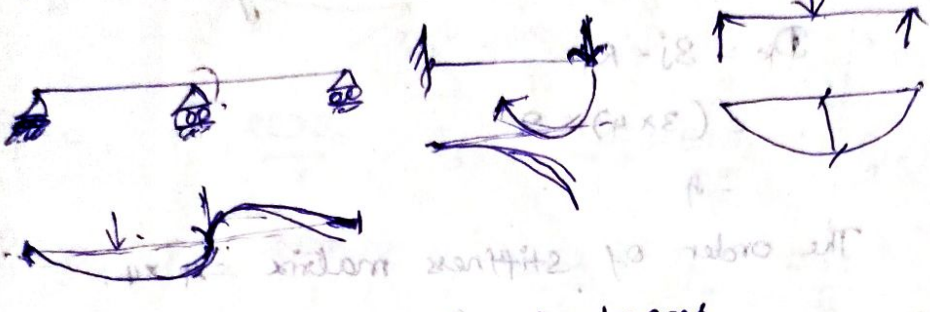
Unit-2 Stiffness Matrix Method

* Suitable for analysing structures by computer applications.
* Used to analyze framed structures.
* Primary unknowns displacements in the form of translation and rotation.



$K.I = D_K = 3j - R \Rightarrow$ beams

$D_K = 2j - R \Rightarrow$ Truss

$D_K = 3j - (m+r) \Rightarrow$ Frames

Stiffness of the Spring:

Load required to produce unit displacement is called Stiffness.

$[S] = [F]^{-1}$ Flexibility = $\frac{\text{Displacement}}{\text{unit force}}$.

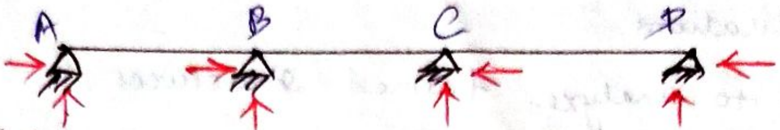
Stiffness = $\frac{\text{Force}}{\text{unit displacement}}$

Support condition	Moment Near end	Far end
Simply supported	$\frac{4EI}{l}$	$\frac{2EI}{l}$

Properties of Stiffness Matrix:

1. It is a square matrix
2. It is a symmetric matrix
3. The diagonal elements are always positive
4. The values of a diagonal elements are high when compared to remaining elements.

1. Determine the stiffness matrix of the beam as shown in figure.



Solution:

$R = 8$

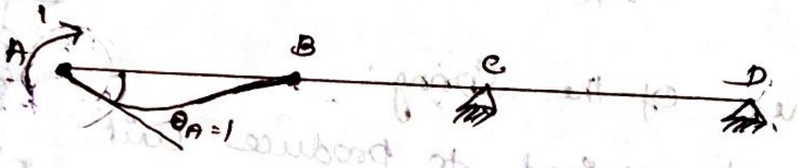
Step 1: Kinematic Indeterminacy:

$$D_k = 3j - R$$
$$= (3 \times 4) - 8$$
$$= 4$$

The order of stiffness matrix $= 4 \times 4$.

Step 2: Stiffness matrix.

For $[k_1]$ apply unit rotation at ①, and restrain 2, 3, 4. $\theta_A = 1$ $\quad \theta_B = \theta_C = \theta_D = 0$.



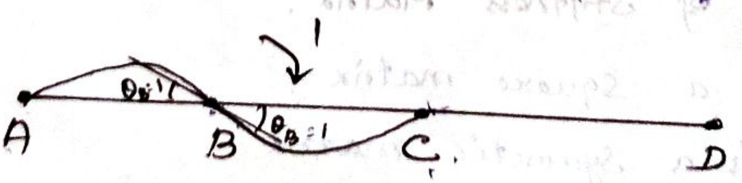
$$P_1 = \frac{4EI\theta_A}{l_{AB}}, \quad P_2 = \frac{2EI\theta_A}{l_{BA}}, \quad P_3 = P_4 = 0.$$

$\theta_A = 1$, $l_{AB} = l_{BC} = l_{CD} = l$

$$P_1 = \frac{4EI}{l}, \quad P_2 = \frac{2EI}{l}, \quad P_3 = P_4 = 0.$$

$$[k_1] = \begin{bmatrix} 4EI/l \\ 2EI/l \\ 0 \\ 0 \end{bmatrix}$$

For k_2 $\quad \theta_B = 1$, $\quad \theta_A = \theta_C = \theta_D = 0$.



$$P_1 = \frac{2EI\theta_B}{l}, \quad P_2 = \frac{4EI\theta_B}{l} + \frac{4EI\theta_B}{l}, \quad P_3 = \frac{2EI\theta_B}{l}, \quad P_4 = 0$$

$$P_1 = \frac{2EI}{l}, \quad P_2 = \frac{8EI}{l}, \quad P_3 = \frac{2EI}{l}, \quad P_4 = 0.$$

$$[k_2] = \begin{bmatrix} 2EI/l \\ 8EI/l \\ 2EI/l \\ 0 \end{bmatrix}$$

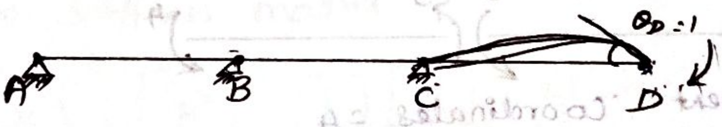
For k_3, $\theta_C = 1$, $\theta_A = \theta_B = \theta_D = 0$.

$P_1 = 0$, $P_2 = \frac{2EI\theta_c}{l}$, $P_3 = \frac{4EI\theta_c}{l} + \frac{4EI\theta_c}{l}$, $P_4 = \frac{2EI\theta_c}{l}$

$P_1 = 0$, $P_2 = \frac{2EI}{l}$, $P_3 = \frac{8EI}{l}$, $P_4 = \frac{2EI}{l}$.

$$[k_3] = \begin{bmatrix} 0 \\ 2EI/l \\ 8EI/l \\ 2EI/l \end{bmatrix}$$

For k_4 $\theta_D = 1$, $\theta_A = \theta_B = \theta_C = 0$



$P_1 = 0$, $P_2 = 0$, $P_3 = \frac{2EI\,\theta_D}{l}$, $P_4 = \frac{4EI.\theta_D}{l}$.

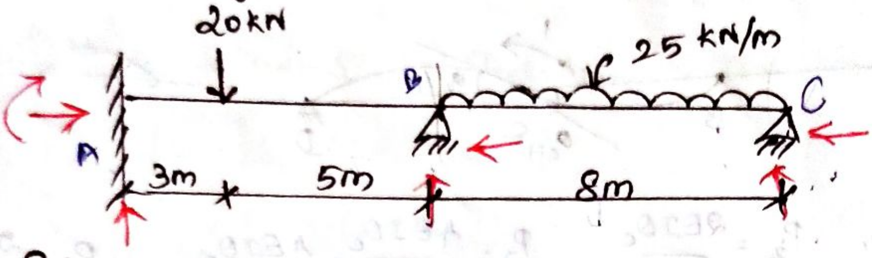
$P_1 = 0$, $P_2 = 0$, $P_3 = \frac{2EI}{l}$, $P_4 = \frac{4EI}{l}$.

$$k_4 = \begin{bmatrix} 0 \\ 0 \\ 2EI/l \\ 4EI/l \end{bmatrix}$$

$$k = \begin{bmatrix} 4EI/l & 2EI/l & 0 & 0 \\ 2EI/l & 8EI/l & 2EI/l & 0 \\ 0 & 2EI/l & 8EI/l & 2EI/l \\ 0 & 0 & 2EI/l & 4EI/l \end{bmatrix}$$

$$k = \frac{EI}{l}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 8 & 2 & 0 \\ 0 & 2 & 8 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

2. A two span continuous beam is fixed at A and hinged over the supports B and C, AB = 8m and BC = 8m. The moment of inertia is constant throughout. It is loaded as shown in fig. Analyse the beam by stiffness matrix method.



R = 7

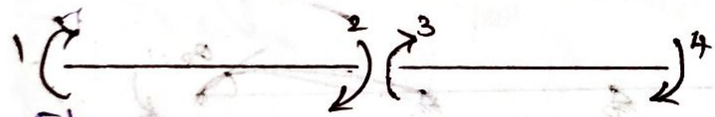
Soln:

Step-1: kinematic indeterminacy:

$$D_K = 3j - R$$
$$= (3\times3) - 7$$
$$= 2$$

System co-ordinates = 2 (B, C)

Element co-ordinates:



Element coordinates = 4

β matrix = 4×2

Fixed End Moments:

$$MF_{AB} = -\frac{wab^2}{l^2} = -\frac{20\times3\times5^2}{8^2} = -23.43 \text{ kNm}$$

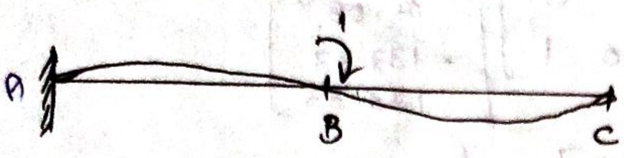
$$MF_{BA} = \frac{wa^2b}{l^2} = \frac{20\times3^2\times5}{8^2} = 14.06 \text{ kNm}$$

$$MF_{BC} = -\frac{wl^2}{12} = -\frac{25\times8^2}{12} = -133.33 \text{ kNm}$$

$$MF_{CB} = \frac{wl^2}{12} = \frac{25\times8^2}{12} = 133.33 \text{ kNm.}$$

$$[P^e] = \begin{bmatrix} -23.43 \\ 14.06 \\ -133.33 \\ 133.33 \end{bmatrix}$$

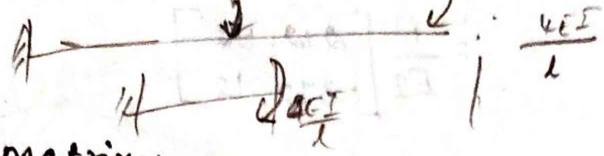
β matrix :



$$\beta_1 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

$$\beta_2 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\beta = \begin{bmatrix}0 & 0\\1 & 0\\1 & 0\\0 & 1\end{bmatrix}$$



Element Stiffness Matrix :

$$[K_1] = \begin{bmatrix} 4EI/8 & 2EI/8 \\ 2EI/8 & 4EI/8 \end{bmatrix} = EI \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}$$

$$K_2 = K_1 \qquad K_n = \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

System Stiffness matrix :

$$\frac{2EI}{8}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$[K] = [\beta]^T [k][\beta]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} EI \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\frac{4EI}{l}$

$\frac{2EI}{l}$

$$= EI \begin{bmatrix} 0.25 & 0.5 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= EI \begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A.$$

$$[K]^{-1} = \frac{1}{EI} \begin{bmatrix} 1.143 & -0.571 \\ -0.571 & 2.28 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

System displacement :

$$[U] = [K]^{-1} \{ f^f - f^o \}$$

$$= \frac{EI}{4} \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \quad f^o = [\beta]^T [P^o] \quad \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

$$f^{o} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -23.43 \\ 14.06 \\ -133.33 \\ 133.33 \end{bmatrix}$$

$$f^{o} = \begin{bmatrix} -119.33 \\ 133.33 \end{bmatrix} \qquad f^{J} = 0$$

$$U = \frac{1}{EI} \begin{bmatrix} 1.143 & -0.571 \\ -0.571 & 2.28 \end{bmatrix} \begin{bmatrix} 119.33 \\ -133.33 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 212.52 \\ -372.12 \end{bmatrix}$$

Element displacement (δ):

$$(\delta) = \{B\}\{U\}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 212.52 \\ -372.12 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 0 \\ 212.52 \\ 212.52 \\ -372.12 \end{bmatrix}$$

Element forces (p')

$$p' = [k][\delta]$$

$$= EI \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 2.5 & 0.5 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 0 \\ 212.52 \\ 212.52 \\ -372.12 \end{bmatrix}$$

$$= \begin{bmatrix} 53.13 \\ 106.26 \\ 13.23 \\ -132.93 \end{bmatrix}$$

Final forces (p^{f}):

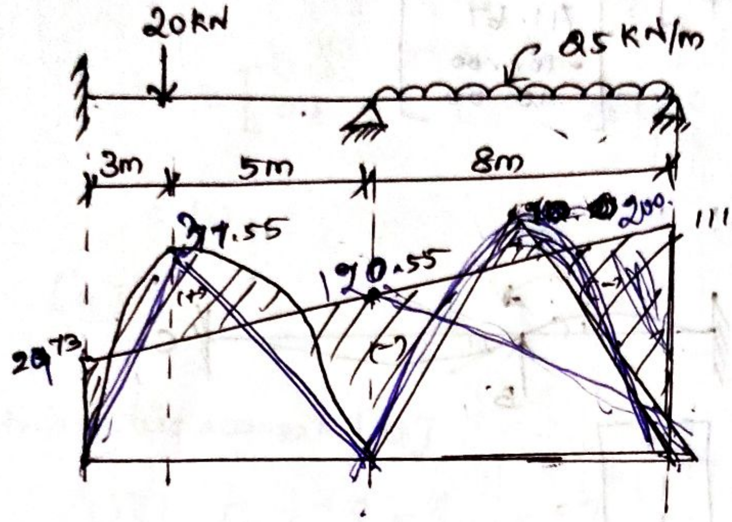
$$p^{f} = [p^{o}] + [p']$$

$$= \begin{bmatrix} -23.43 \\ 14.06 \\ -133.33 \\ 133.33 \end{bmatrix} + \begin{bmatrix} 53.13 \\ 106.26 \\ 13.23 \\ -133 \end{bmatrix} = \begin{bmatrix} 29.7 \\ 120.32 \\ -120.1 \\ 0 \end{bmatrix}$$

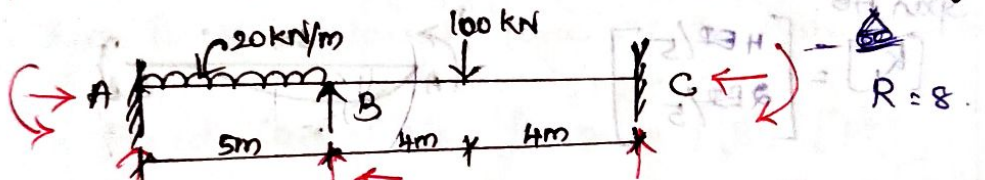
Maximum Bending moments :

$$M_{AB} = \frac{Wab}{l} = \frac{20 \times 3 \times 5}{8} = 37.5 \text{ kNm}$$

$$M_{BC} = \frac{Wl^2}{8} = \frac{25 \times 8^2}{8} = 200 \text{ kNm.}$$



3. Analyse the continuous beam as shown in figure by Stiffness method and draw the bending moment diagram.



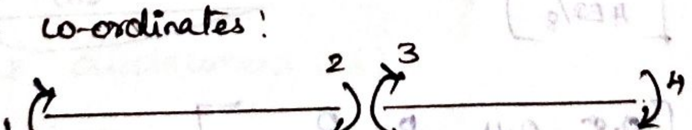
Soln : Both ends are fixed, so the center will be a pin jointed only.

Kinematic Indeterminacy :

$$D_k = 3j - R$$
$$= (3 \times 3) - 8$$
$$= 1$$

System co-ordinate = 1 (B)

Element co-ordinates :



Fixed End moments :

$$MF_{AB} = -\frac{Wl^2}{12} = -\frac{20 \times 5^2}{12} = -41.67 \text{ kNm.}$$

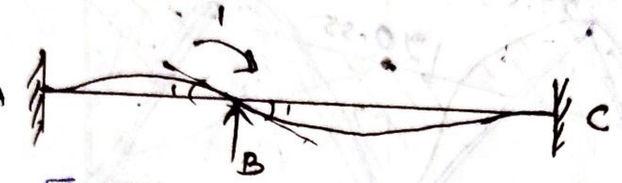
$$MF_{BA} = \frac{Wl^2}{12} = \frac{20 \times 5^2}{12} = 41.67 \text{ kNm}$$

$$MF_{BC} = -\frac{Wl}{8} = \frac{100 \times 8}{8} = -100 \text{ kNm.}$$

$$MF_{CB} = \frac{Wl}{8} = \frac{100 \times 8}{8} = 100 \text{ kNm}$$

$$[P^{\circ}] = \begin{bmatrix} -41.67 \\ 41.67 \\ -100.00 \\ 100.00 \end{bmatrix}$$

β matrix :

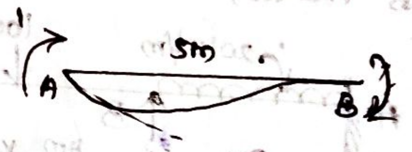


$$\beta = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

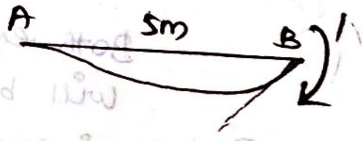
Element stiffness matrix :

Span AB :

$$[k_1] = \begin{bmatrix} 4EI/5 \\ 2EI/5 \end{bmatrix}$$

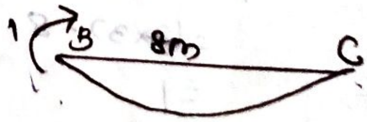


$$k_2 = \begin{bmatrix} 2EI/5 \\ 4EI/5 \end{bmatrix}$$

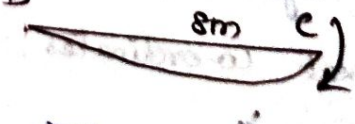


Span BC :

$$k_3 = \begin{bmatrix} 4EI/8 \\ 2EI/8 \end{bmatrix}$$



$$k_4 = \begin{bmatrix} 2EI/8 \\ 4EI/8 \end{bmatrix}$$



$$[K] = \begin{bmatrix} 0.8 & 0.4 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0.50 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

System Stiffness matrix :

$$[K] = [B]^T [k] [B]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} EI \begin{bmatrix} 0.8 & 0.4 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.4 & 0.8 & 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= 1.3 EI$$

$$[K]^{-1} = \frac{1}{1.3 EI}$$

System displacement $[U]$:

$$[U] = [K]^{-1} \{ F^f - F^o \}$$

$[F^f]$ forces applied at system co-ordinates $= 0$.

[∴ There is no external force at the system coordinates]

F^o = Fixed Coordinate forces $= [B]^T [P^o]$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -41.67 \\ 41.67 \\ -100 \\ 100 \end{bmatrix}$$

$$= [-58.33]$$

$$U = \frac{1}{1.3 EI} \left[0 - (-58.33) \right]$$

$$= \frac{58.33}{1.3 EI} = \frac{44.86}{EI}$$

Element displacement (δ) :

$$\delta = [B][U]$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \frac{44.86}{EI} = \frac{1}{EI} \begin{bmatrix} 0 \\ 44.86 \\ 44.86 \\ 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 0 \\ 44.86 \\ 44.86 \\ 0 \end{bmatrix}$$

Element forces [P'] :

$$[P'] = [k][\delta]$$

$$= EI \begin{bmatrix} 0.8 & 0.4 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 0 \\ 44.86 \\ 44.86 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 17.94 \\ 35.88 \\ 22.43 \\ 11.215 \end{bmatrix}$$

Final forces (pf):

$$pf = [P^0] + [P']$$

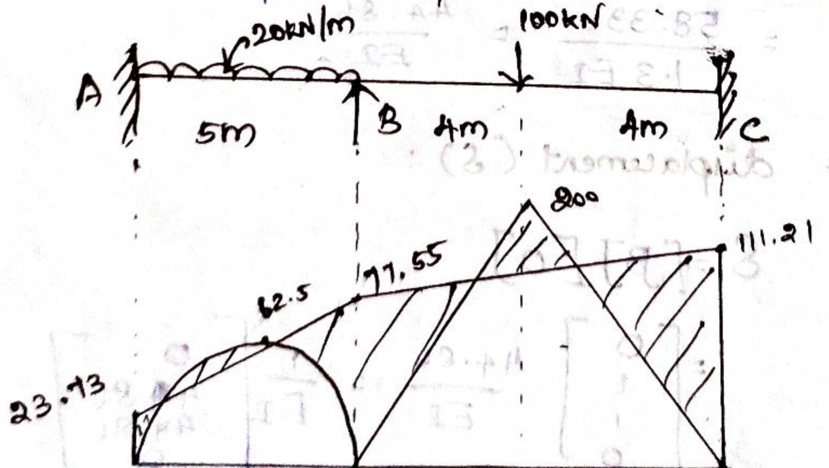
$$= \begin{bmatrix} -41.67 \\ 41.67 \\ -100 \\ 100 \end{bmatrix} + \begin{bmatrix} 17.94 \\ 35.88 \\ 22.43 \\ 11.215 \end{bmatrix}$$

$$= \begin{bmatrix} -23.73 \\ 77.55 \\ -77.55 \\ 111.21 \end{bmatrix}$$

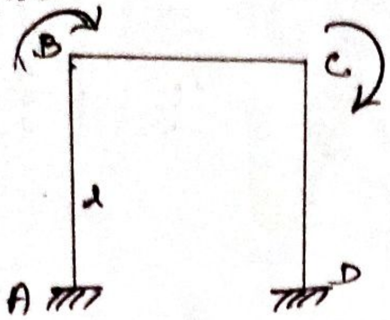
Maximum Bending Moment :

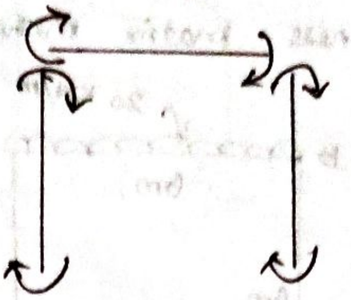
$$M_{AB} = \frac{wl^2}{8} = \frac{20 \times 5^2}{8} = 62.5 \text{ kNm}$$

$$M_{BC} = \frac{Wl}{4} = \frac{100 \times 8}{4} = 200 \text{ kNm}.$$



4. Generate the β matrix for the Portal frame as shown in figure.





Element coordinates

Soln :

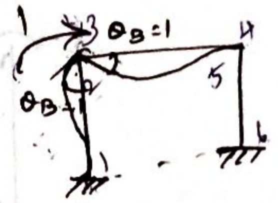
Kinematic indeterminacy :

$$D_K = 3j - (m+r)$$

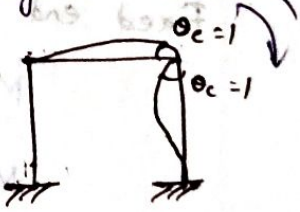
Here there is 2 system coordinates and 6 element coordinates.

Apply unit rotation at first system coordinate

$$\beta = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



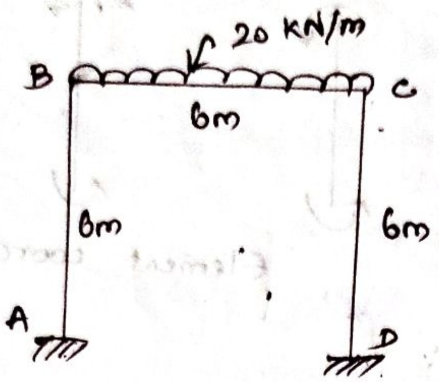
Apply unit rotation at Second system coordinate



$$\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

5. Find Bending moment of the given frame by Stiffness matrix method.

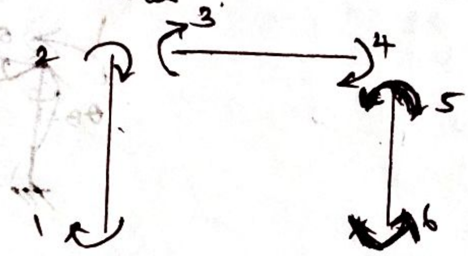


Soln :

Unknown displacements $= \theta_B, \theta_C$.

System coordinates = 2 (B and C)

Element coordinates :



Fixed end moments :

$$MF_{AB} = MF_{BA} = 0$$

$$MF_{BC} = \frac{-wl^2}{12} = \frac{-20 \times 6^2}{12}$$

$$= -60 \text{ kNm}$$

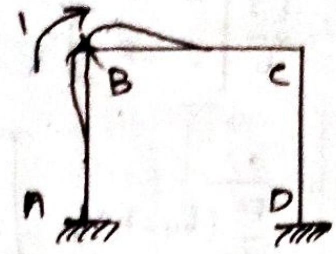
$$MF_{CB} = \frac{wL^2}{12} = \frac{20 \times 6^2}{12} = 60 \text{kNm}$$

$$MF_{CD} = MF_{DC} = 0$$

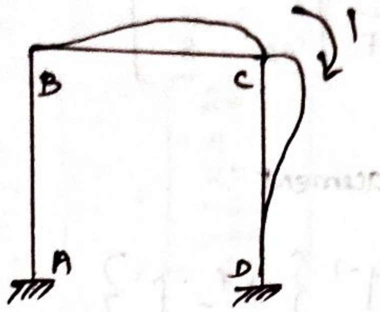
$$P^0 = \begin{bmatrix} 0 \\ 0 \\ -60 \\ 60 \\ 0 \\ 0 \end{bmatrix}$$

β Matrix :

$$\beta_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\beta_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$



$$\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Element ~~Syst~~ Stiffness Matrix !

$$K_1 = \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$K_2 = \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \qquad K_3 = \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$k = \frac{EI}{6}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

System Stiffness Matrix :

$$K = [\beta]^T [k][\beta]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \frac{EI}{6}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$K = \frac{EI}{6}\begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix}$$

$$K^{-1} = \frac{1}{\frac{EI}{6}(64-4)}\begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix}$$

$$= \frac{1}{10EI}\begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix}$$

System displacement:

$$U = [K]^{-1}\{F^f - f^o\}$$

$$f^o = [B]^T[P^o]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -60 \\ 60 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -60 \\ 60 \end{bmatrix}$$

$$U = \frac{1}{10EI}\begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -60 \\ 60 \end{bmatrix}\right\}$$

$$= \frac{1}{EI}\begin{bmatrix} 60 \\ -60 \end{bmatrix}$$

Element displacement:

$$\delta = [B][U]$$

$$= \underset{(6\times 2)}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \frac{1}{EI}\underset{(2\times 1)}{\begin{bmatrix} 60 \\ -60 \end{bmatrix}}$$

$$= \frac{1}{EI}\underset{(6\times 1)}{\begin{bmatrix} 0 \\ 60 \\ 60 \\ -60 \\ -60 \\ 0 \end{bmatrix}}$$

Element force (p') :

$$p = [k][\delta]$$

$$= \frac{EI}{6}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}_{(6\times6)} \begin{bmatrix} 0 \\ 60 \\ 60 \\ -60 \\ -60 \\ 0 \end{bmatrix}_{(6\times1)} \frac{1}{EI}$$

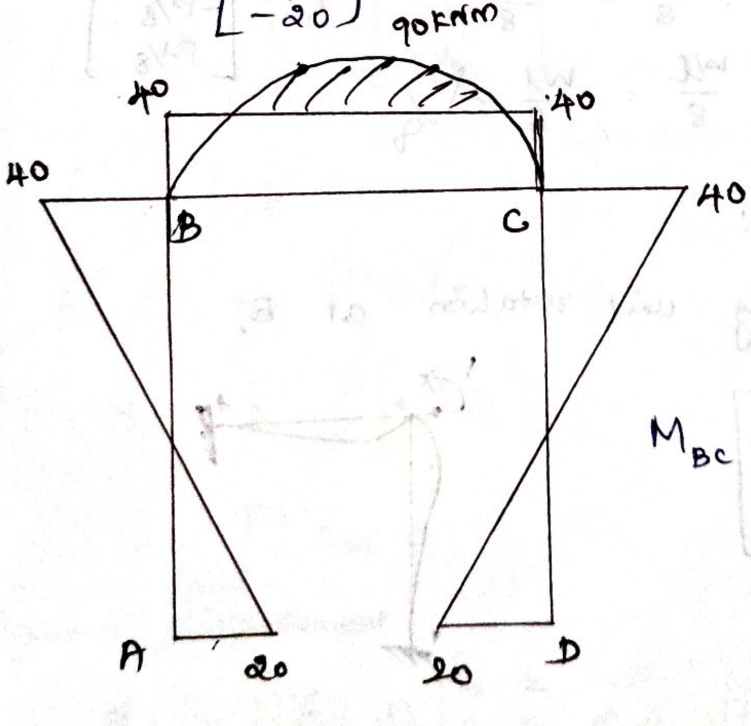
$$= \frac{1}{6}\begin{bmatrix} 120 \\ 240 \\ 120 \\ -120 \\ -240 \\ -120 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 20 \\ -20 \\ -40 \\ 20 \end{bmatrix}$$

Final force (p^f) :

$$p^f = p' + p^\circ$$

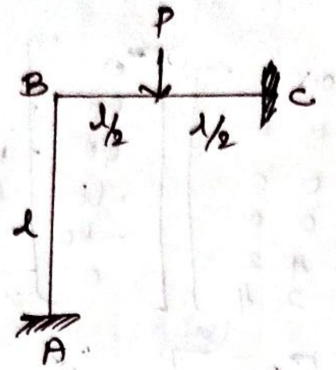
$$= \begin{bmatrix} 20 \\ 40 \\ 20 \\ -20 \\ -40 \\ -20 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -60 \\ 60 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ 40 \\ -40 \\ 40 \\ -40 \\ -20 \end{bmatrix} \text{ kNm.}$$



$$M_{BC} = \frac{Wl}{8} = \frac{20\times6}{8} = 90 \text{ kNm.}$$

6. Analyse the frame as shown in figure by Stiffness method.



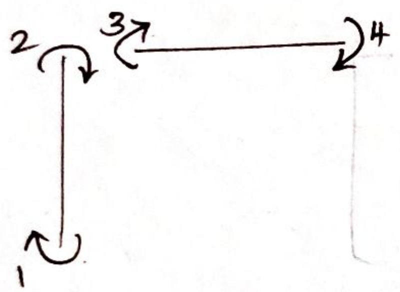
<u>Soln</u>:

System coordinates:

Unknown displacements $= \theta_B$

system coordinate $= 1$ (B)

Element Coordinates:



Fixed end Moments:

$MF_{AB} = MF_{BA} = 0.$

$MF_{BC} = -\frac{Wl}{8} = -\frac{Pl}{8}$

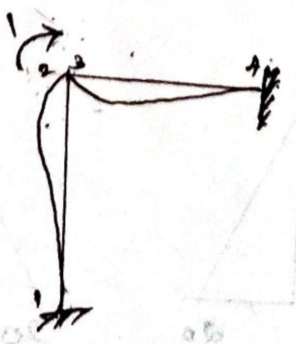
$MF_{CB} = \frac{Wl}{8} = \frac{Wl}{8} = Pl/8$

$$p^{\circ} = \begin{bmatrix} 0 \\ 0 \\ -Pl/8 \\ Pl/8 \end{bmatrix}$$

β matrix:

Applying unit rotation at B,

$$\beta = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$



Element stiffness matrix :

$$k_1 = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \qquad k_2 = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$k = \frac{EI}{l}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

System Stiffness matrix :

$$K = [B]^T[k][B]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \frac{EI}{l}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{8EI}{l}$$

$$K^{-1} = \frac{l}{8EI}$$

System displacement :

$$U = [K]^{-1}\{f^f - f^0\}$$

$$f^0 = [B]^T[P^0]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -Pl/8 \\ Pl/8 \end{bmatrix}$$

$$f^0 = -Pl/8$$

$$f^f - f^0 = 0 - (-Pl/8) = Pl/8.$$

$$U = l/8EI\,[Pl/8]$$

$$= Pl^2/64EI$$

Element displacement :

$$\delta = [B][u]$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} Pl^2/64EI$$

Element ~~displacement~~ force :

$$P = [k][\delta] = \frac{EI}{l}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\frac{Pl^2}{64EI}$$

$$P' = \frac{Pl}{64}\begin{bmatrix}2\\4\\4\\2\end{bmatrix}$$

Final forces :

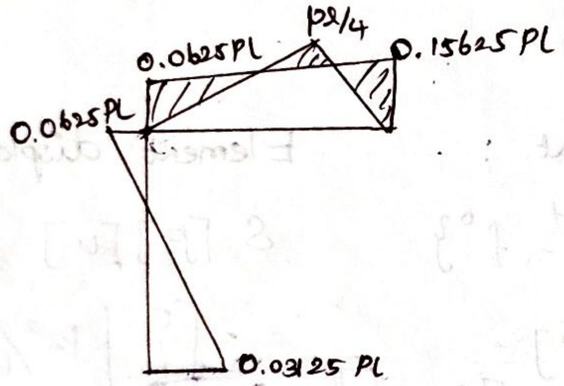
$$P^f = P^o + P'$$

$$= \begin{bmatrix}0\\0\\-Pl/8\\Pl/8\end{bmatrix} + \frac{Pl}{64}\begin{bmatrix}2\\4\\4\\2\end{bmatrix}$$

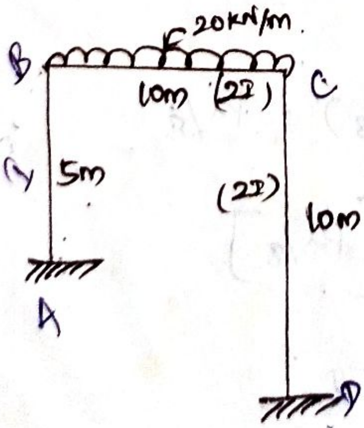
$$= Pl\begin{bmatrix}0.031\\0.0625\\-0.0625\\0.15625\end{bmatrix}$$

Maximum Bending moment :

$$M_{BC} = \frac{Wl}{4} = \frac{Pl}{4} \cdot 0.25Pl$$

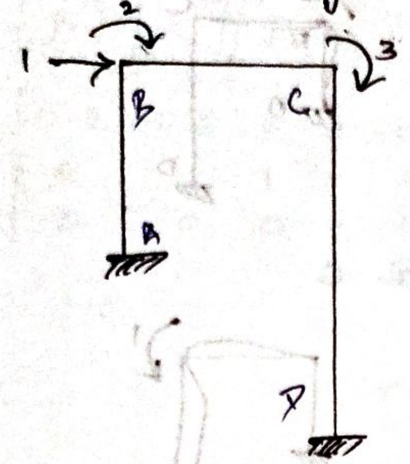


7. Analyse the frame as shown in figure by Stiffness matrix method.



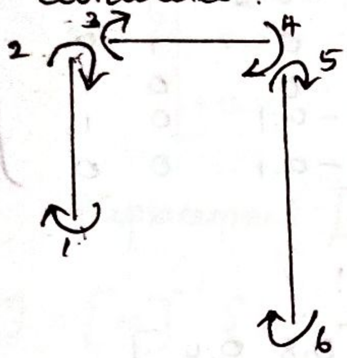
Soln :

i) Kinematic indeterminacy :



This structure is kinematically indeterminate to 3 degree $(\theta_B, \delta_B, \theta_C)$

Element coordinates :



Element coordinates = 6.

Fixed end moments :

$$MF_{AB} = MF_{BA} = MF_{CD} = MF_{DC} = 0.$$

$$MF_{BC} = -\frac{Wl^2}{12} = -\frac{20\times10^2}{12} = -166.67 \text{ KNm}.$$

$$MF_{CB} = \frac{Wl^2}{12} = \frac{20\times10^2}{12} = 166.67 \text{ KNm}.$$

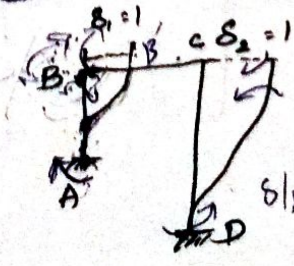
$$P^0 = \begin{bmatrix} 0 \\ 0 \\ -166.67 \\ 166.67 \\ 0 \\ 0 \end{bmatrix}$$

β matrix :

Applying unit displacement @ B.

$$\beta_1 = \begin{bmatrix} -1/5 \\ -1/5 \\ 0 \\ 0 \\ -1/10 \\ -1/10 \end{bmatrix}$$

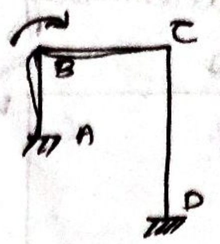
$\delta/l = 1/5$



$\delta/l = 1/10$

Applying unit rotation @ B,

$$\beta_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Applying unit rotation @ C

$$\beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\beta_1 = \begin{bmatrix} -0.2 \\ -0.2 \\ 0 \\ 0 \\ -0.1 \\ -0.1 \end{bmatrix} \qquad \beta = \begin{bmatrix} -0.2 & 0 & 0 \\ -0.2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.1 & 0 & 1 \\ -0.1 & 0 & 0 \end{bmatrix}$$

Element Stiffness Matrix :

$$k_1 = \frac{EI}{5}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$$

$$k_2 = \frac{E \times 2I}{10}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$$

$$k_3 = \frac{E \times 2I}{10}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$$

$$[K] = EI\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.4 & 0.8 \end{bmatrix}$$

System stiffness matrix:

$$K = [\beta]^T [k] [\beta]$$

$$= \begin{bmatrix} -0.2 & -0.2 & 0 & 0 & -0.1 & -0.1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}_{(3\times 6)} EI \begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.4 & 0.8 \end{bmatrix}_{(6\times 6)}$$

$$\times \begin{bmatrix} -0.2 & 0 & 0 \\ -0.2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.1 & 0 & 1 \\ -0.1 & 0 & 0 \end{bmatrix}_{(6\times 3)}$$

$$= EI \begin{bmatrix} 0.12 & -0.24 & -0.12 \\ -0.24 & 1.6 & 0.4 \\ -0.12 & 0.4 & 1.6 \end{bmatrix}_{(3\times 3)}$$

System displacement $[U]$:

$$[U] = [K]^{-1} \{ f^t - f^o \}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$[K]^{-1} = \frac{1}{EI} \begin{bmatrix} 12.25 & 1.715 & 0.490 \\ 1.715 & 0.906 & -0.098 \\ 0.490 & -0.098 & 0.686 \end{bmatrix}$$

$$F^o = [\beta]^T [P^o]$$

$$F^o = \begin{bmatrix} 0 \\ -166.67 \\ 166.67 \end{bmatrix}$$

$$[U] = \frac{1}{EI} \begin{bmatrix} 12.25 & 1.715 & 0.490 \\ 1.715 & 0.906 & -0.098 \\ 0.490 & -0.098 & 0.686 \end{bmatrix} \begin{bmatrix} 0 \\ 166.67 \\ -166.67 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 204.17 \\ 167.33 \\ -130.66 \end{bmatrix}$$

Element displacement (δ):

$$\delta = [\beta][U]$$

$$= \begin{bmatrix} -0.2 & 0 & 0 \\ -0.2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.1 & 0 & 1 \\ -0.1 & 0 & 0 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 204.17 \\ 167.33 \\ -130.66 \end{bmatrix}$$

$$\delta = \frac{1}{EI}\begin{bmatrix} -40.83 \\ 126.496 \\ 167.33 \\ -130.66 \\ -151.07 \\ -20.47 \end{bmatrix}$$

Element Forces:

$$P' = [k][\delta]$$

$$= EI\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.4 & 0.8 \end{bmatrix}_{(6\times 6)} \frac{1}{EI}\begin{bmatrix} -40.83 \\ 126.496 \\ 167.33 \\ -130.66 \\ -151.07 \\ -20.47 \end{bmatrix}_{(6\times 1)}$$

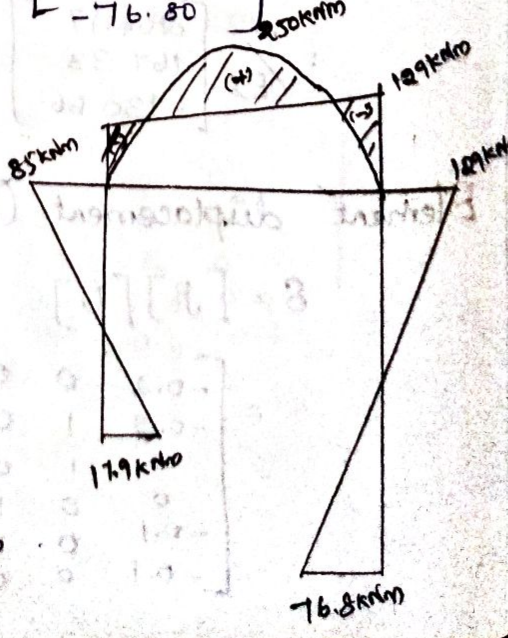
$$= \begin{bmatrix} 17.93 \\ 84.86 \\ 81.60 \\ -37.59 \\ -129.04 \\ -76.80 \end{bmatrix}$$

Final Forces:

$$P^f = P^0 + P'$$

$$= \begin{bmatrix} 0 \\ 0 \\ -166.67 \\ 166.67 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 17.93 \\ 84.86 \\ 81.60 \\ -37.59 \\ -129.04 \\ -76.80 \end{bmatrix}$$

$$= \begin{bmatrix} 17.9 \\ 85 \\ -85 \\ 129 \\ -129 \\ -76.80 \end{bmatrix}$$



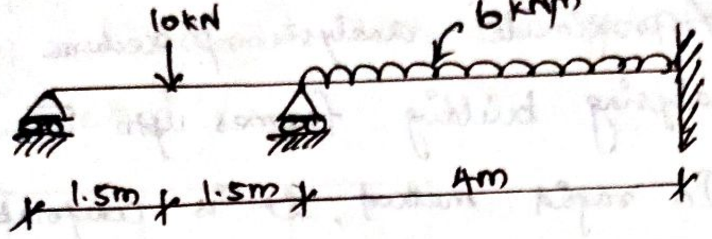
Maximum Bending moment:

$M_{AB} = M_{CD} = 0$

$$M_{BC} = \frac{Wl^2}{8} = \frac{20\times 10^2}{8} = 250\ kNm$$

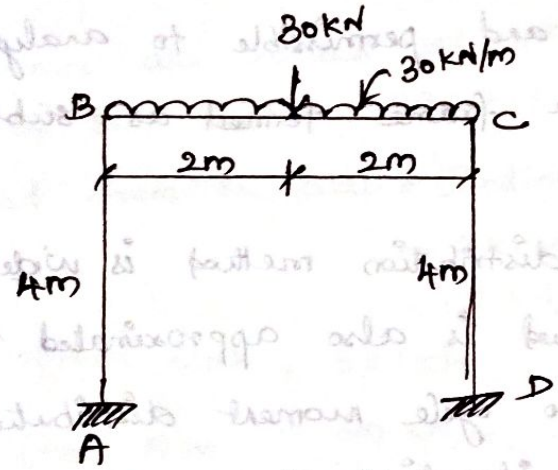
Assignment - 2 Problems :

1. Analyse the continuous beam as shown in figure by stiffness matrix method and sketch the BMD.



EI = constant

2. Analyse the portal frame ABCD shown in figure by Stiffness matrix method and also sketch the BMD.



EI = Constant