Linear PDE's of Second order with constant Coefficients

Homogeneous Linear PDE

$$\left[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n\right] z = f(x, y)$$

where $D \equiv \frac{\partial}{\partial x}$; $D' \equiv \frac{\partial}{\partial y}$

i.e., $f(D, D') z = F(x, y)$

$\therefore$ The soln of ② is $z = CF + PI$

C.F $\rightarrow$ Complementary Function

P.I $\rightarrow$ Particular Integral.

To find CF:

Replace $D \to m$
$D' \to 1$ m given.

Case (i): roots are real and distinct

$m_1, m_2, \ldots$

$$z = f_1(y + m_1 x) + f_2(y + m_2 x) + \cdots + f_n(y + m_n x)$$

Case (ii): any two roots are equal

$$z = f_1(y + mx) + x f_2(y + mx) + f_3(y + m_3 x) + \cdots + f_n(y + m_n x)$$

Case (iii): all roots are equal.

$$z = f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx) + \cdots x^n f_n(y + m_n x)$$

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$+ \cdots - x + n | y + m, x)$

Particular Integral:

Rules for finding P.I

$$f(D, D')z = F(x, y)$$

$$P.I = \frac{1}{f(D, D')} F(x, y)$$

(i) $F(x, y) = e^{ax+by}$

Replace $D \to a$ and $D' \to b$.

$$P.I = \frac{1}{f(D, D')} e^{ax+by}$$

$$= \frac{1}{f(a, b)} e^{ax+by}, \text{ if } f(a, b) \neq 0.$$

(ii) $F(x, y) = x^m y^n$

$$P.I = \frac{1}{f(D, D')} x^m y^n.$$

$$= [f(D, D')]^{-1} x^m y^n.$$

Expand $f(D, D')^{-1}$ using Binomial Theorem.

1) Solve $[D^2 - 5DD' + 6D'^2]z = 0$.

2) Solve $[D^2 - 6DD' + 9D'^2]z = 0$.

3) Solve $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$.

1) Solve $[D^2 - 5DD' + 6D'^2]z = 0$

Soln: The auxiliary eqn is given by

Replace $D \to m$ and $D' \to 1$

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$-3 \quad -2$

$$m^2 - 5m + 6 = 0.$$

$$(m-3)(m-2) = 0$$

$$m = 3, 2$$

$$m_1 = 3 \text{ and } m_2 = 2$$

The roots are real and distinct

$\therefore$ The complementary function is given by

$$C.F. \; z = f_1(y + m_1 x) + f_2(y + m_2 x)$$

$$z = f_1(y + 3x) + f_2(y + 2x).$$

$(D^2 - 6DD' + 9D'^2)z = 0$

$D \rightarrow m \quad D' \rightarrow 1$

$m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m = 3, 3$

roots are real & equal

C.F

$z = f_1(y+3x) + x f_2(y+3x)$.

3) $2D^2 + 5DD' + 2D'^2 = 0$.

$D \rightarrow m \quad D' \rightarrow 1$

$2m^2 + 5m + 2 = 0$.

~~2m² + 4m~~

$\left(m + \frac{4}{2}\right)\left(m + \frac{1}{2}\right) = 0$

$(m+2)(2m+1) = 0$

$m_1 = -2$ and $m_2 = \frac{-1}{2}$.

roots are real & distinct

C.F $\quad z = f_1(y - 2x) + f_2\left(y - \frac{1}{2}x\right)$.

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$-2 \quad -2$

4

$-\frac{4}{2} \quad -\frac{1}{2}$