Type-I $F(p,q)=0$

Ans: Let us assume, $z = ax+by+c$ — ② be the Soln of $F(p,q)=0$ — ①

(*) Put $p=a$ and $q=b$ in ①

(*) Find the value of b in terms of a.

$z = ax + \phi(a)\cdot y + c$ is the C.I.

(*) no S.I

(*) put $c = f(a)$ in C.I. and diff w.r.to. a

(*) Eliminate 'a' from above two eqns, is the G.I.

1) Solve $\sqrt{p}+\sqrt{q}=1$.

Soln:

$$\sqrt{p}+\sqrt{q}=1 \quad \text{—— ①}$$

Let the solution of ① be $z=ax+by+c$ —— ②

C.I: put $p=a$ and $q=b$ in ①

$$\sqrt{a}+\sqrt{b}=1$$

$$\sqrt{b}=1-\sqrt{a}$$

$$b=(1-\sqrt{a})^2 \quad \text{—— ③}$$

Sub (3) in (2).

$$z = ax + (1-\sqrt{a})^2 y + c \quad \text{——— (4)}$$

is the Complete Integral.

G.I:

put $c = f(a)$ in (4)

$$z = ax + (1-\sqrt{a})^2 y + f(a) \quad \text{——— (5)}$$

Diff w. r. to 'a' partially

$$\frac{\partial z}{\partial a} \Rightarrow 0 = x + 2(1-\sqrt{a})\left(-\frac{1}{2}a^{-1/2}\right)y + f'(a)$$

$$x + \frac{-2}{2}\frac{(1-\sqrt{a})}{a^{1/2}} y + f'(a) = 0.$$

$$x - \frac{(1-\sqrt{a})}{a^{1/2}} y + f'(a) = 0 \quad \text{——— (6)}$$

Eliminating a (5) & (6) gives the general Integral.

2) Solve $p+q = pq$ ——— (1) $F(p,q)=0$.

Soln: Let $z = ax+by+c$ ——— (2) be the solution of (1)

C.I: put $p=a$ and $q=b$ in (1).

$$a+b = ab$$

$$b - ab = -a$$

$$b(1-a) = -a$$

$$b = \frac{-a}{(1-a)} \quad \text{——— (3).}$$

Sub (3) in (2),

$$z = ax - \frac{a}{(1-a)} y + c \quad \text{——— (4).}$$

is the complete Integral.

G.I: put $c = f(a)$ in (4)

$$z = ax - \frac{a}{(1-a)} y + f(a) \quad \text{——— (5).}$$

Diff w.r.to. a par, $0 = x - y\left[\frac{(1-a) - a(-1)}{(1-a)^2}\right] + f'(a)$

$$x - \frac{y}{(1-a)^2} + f'(a) = 0 \quad \text{——— (6)}$$

Eliminating 'a' from (5) & (6) gives the general Integral.

$$= \frac{-x-2y}{3}$$

Type-II **Clairaut's form**

* $z = px + qy + f(p,q)$ —— (1)

put $p = a$ and $q = b$ in (1), $z = ax + by + f(a,b)$

Find values of a and b, by diff par. w. r. to a & b.

1) $z = px + qy + p^2 + pq + q^2$ —— (1)

This is of the form $z = px + qy + f(p,q)$

put $p = a$ and $q = b$ in (1),

$z = ax + by + f(a,b) = ax + by + a^2 + ab + b^2$

C.I is $z = ax + by + a^2 + ab + b^2$ —— (2).

Singular Integral:

Diff (2) partially w. r. to 'a'

$0 = x + 0 + 2a + b + 0$

$x + 2a + b = 0 \Rightarrow a = -\frac{(b+x)}{2}$ —— (3)

Diff (2) partially w. r. to 'b'

$0 = 0 + y + 0 + a + 2b$

$y + a + 2b = 0 \Rightarrow y - \frac{(b+x)}{2} + 2b = 0$

$y - \frac{b}{2} - \frac{x}{2} + 2b = 0$

$\frac{3b}{2} = \frac{x}{2} - y \Rightarrow \boxed{b = \frac{x-2y}{3}}$ —— (4)

sub (4) in (3)

$a = -\frac{1}{2}\left[\frac{x-2y}{3} + x\right] = -\frac{1}{2}\left[\frac{4x-2y}{3}\right] = \frac{-2x+y}{3}$

$$\boxed{a = \frac{y-2x}{3}} \quad \text{——— (5)}$$

Sub (5) & (4) in (2), we get

$$z = \left(\frac{y-2x}{3}\right)x + \left(\frac{x-2y}{3}\right)y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

$$= \frac{xy-2x^2}{3} + \frac{xy-2y^2}{3} + \frac{y^2-2xy+4x^2}{9} + \frac{xy-2y^2-2x^2+4xy}{9} + \frac{x^2-4xy+4y^2}{9}$$

$$= \frac{1}{9}\left[3xy - 6x^2 + 3xy - 6y^2 + y^2 - 2xy + 4x^2 + xy - 2y^2 - 2x^2 + 4xy + x^2 - 4xy + 4y^2\right]$$

$$= \frac{1}{9}\left[3xy - 3x^2 - 3y^2\right]$$

$\Rightarrow 3z = xy - x^2 - y^2$ which is the S.I.

(2) Solve $z = px + qy + 2\sqrt{pq}$ ——— (1)

* This is of the form $z = px + qy + f(p,q)$ clairaut's form,

* put $p = a$; $q = b$

C.I $\quad z = ax + by + f(a,b)$

$$z = ax + by + 2\sqrt{ab} \quad \text{——— (2)}$$

S.I Diff (2) w. r. to 'a' & 'b'

$$0 = x + 0 + 2\sqrt{b}\,\frac{1}{2}a^{-1/2}$$

$$x + \frac{\sqrt{b}}{\sqrt{a}} = 0 \Rightarrow \boxed{x = -\sqrt{\frac{b}{a}}} \quad \text{——— (3)}$$

$$0 = 0 + y + 2\sqrt{a}\,\frac{1}{2}b^{-1/2}$$

$$y + \sqrt{\frac{a}{b}} = 0 \Rightarrow \boxed{y = -\sqrt{\frac{a}{b}}} \quad \text{——— (4)}$$

from (3) & (4) $\quad xy = \left(-\sqrt{\frac{b}{a}}\right)\left(-\sqrt{\frac{a}{b}}\right) \Rightarrow \boxed{xy = 1}$ is the S.I

(3) Solve : $(1-x)p+(2-y)q = 3-z$ —— (1)

Soln: This is of the form

$$p - px + 2q - yq - 3 + z = 0$$

$$z = px + qy - p - 2q + 3$$

$$z = px + qy + f(p,q) \text{ (Clairaut's form)}$$

C.I put $p = a$ and $q = b$.

$$z = ax + by + f(a,b)$$

$$z = ax + by - a - 2b + 3 \quad \text{—— (2)}.$$

S.I diff (2) partially w.r. to a and b

$$0 = x + 0 - 1 - 0 + 0 \Rightarrow x = 1 \quad \text{—— (3)}$$

$$0 = 0 + y - 0 - 2 + 0 \Rightarrow y = 2 \quad \text{—— (4)}.$$

sub (3) & (4) in (2)

$$z = a + 2b - a - 2b + 3$$

$\boxed{z = 3}$ which is the req S.I.

(3) Solve $z = px + qy + \sqrt{p^2+q^2+1}$ —— (1)

This is of the form

$$z = px + qy + f(p,q) \text{ (Clairaut's form)}$$

C.I: put $p = a$ and $q = b$.

$$z = ax + by + \sqrt{a^2+b^2+1} \quad \text{—— (2)}.$$

S.I: diff (2) partially w.r. to a and b.

$$(2) \Rightarrow \frac{\partial z}{\partial a} = 0 = x + 0 + \frac{1}{2\sqrt{a^2+b^2+1}}(2a)$$

$$\Rightarrow x = \frac{-a}{\sqrt{a^2+b^2+1}} \quad // \Rightarrow x^2 = \frac{a^2}{a^2+b^2+1} \quad \text{—— (3)}$$

$$(2) \Rightarrow \frac{\partial z}{\partial b} = 0 = 0 + y + \frac{1}{2\sqrt{a^2+b^2+1}}(2b)$$

$$\Rightarrow y = \frac{-b}{\sqrt{a^2+b^2+1}} \quad // \quad y^2 = \frac{b^2}{a^2+b^2+1} \quad \text{—— (4)}$$

Using ③ & ④, $x^2+y^2 = \dfrac{a^2+b^2}{1+a^2+b^2}$

$$1-(x^2+y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$= \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2} = \frac{1}{1+a^2+b^2}$$

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}} \Rightarrow \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}} \quad \text{— ⑤}$$

sub ⑤ in ③ & ④

$$a = \frac{-x}{\sqrt{1-x^2-y^2}} \text{ and } b = \frac{-y}{\sqrt{1-x^2-y^2}} \quad \text{— ⑥}$$

Sub ⑤ & ⑥ in ② we get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow z = (1-x^2-y^2)^{1/2}$$

$$\Rightarrow z^2 = 1-x^2-y^2.$$

$\Rightarrow x^2+y^2+z^2 = 1$ is the S.I.

HW: 1) Solve $z = px+qy+\frac{p}{q}-p$.

2) Find C.I of (i) $pqz = p^2(qx+p^2)+q^2(py+q^2)$ Hint: Convert to clairaut's form.

(ii) $\left(\frac{z}{pq}\right) = \frac{x}{q}+\frac{y}{p}+\sqrt{pq}$

3) Find SI of (i) $z = px+qy+p^2$.

(ii) $(pq-p-q)(z-px-qy) = pq$.

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