UNIT-3

Partial Differential Equations.

Defn: A PDE is an eqn involving a function of two or more variables and some of its partial derivatives.

$p = \frac{\partial z}{\partial x}$; $q = \frac{\partial z}{\partial y}$; $r = \frac{\partial^2 z}{\partial x^2}$; $s = \frac{\partial^2 z}{\partial x \partial y}$; $t = \frac{\partial^2 z}{\partial y^2}$

when $z = f(x, y)$

Defn: An order of a PDE is the order of the highest partial derivative occuring in the equation.

Formation of Partial Differential Equations

(i) By elimination of Arbitary constants

(ii) By elimination of Arbitrary functions.

By Elimination of Arbitrary Constants

Type - I

If number of a.c $\leq$ no. of I.V, use I order PDE

a.c — arbitrary constants

I V — Independent variables.

1) Form a pde by eliminating a.c. for the following functions.

(i) $z = ax + by + a^2 + b^2$

(ii) $z = (x+a)^2 + (y-b)^2$

(iii) $z = (x+a)(y+b)$

(iv) $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$.

Solution: (i) Gn: $z = ax + by + a^2 + b^2$ —— ①

Diff ① partially w.r.to x

$$\frac{\partial z}{\partial x} = p = a$$

Diff ① partially w.r.to y

$$\frac{\partial z}{\partial y} = q = b$$

Substitute p and q in ①.

$z = px + by + a^2 + b^2$. is the req PDE.

(ii) Gn: $z = (x+a)^2 + (y-b)^2$ —— ①

Diff ① partially w.r.to x

$$\frac{\partial z}{\partial x} = p = 2(x+a)(1) + 0$$

$$p = 2(x+a) \Rightarrow (x+a) = \frac{p}{2}$$

Diff ① partially with r. to y

$$\frac{\partial z}{\partial y} = q = 0 + 2(y-b)$$

$$q = 2(y-b) \Rightarrow (y-b) = \frac{q}{2}$$

Sub p and q in ①

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 = \frac{p^2}{4} + \frac{q^2}{4},$$

is the req PDE

(iii) Gn: $z=(x+a)(y+b)$ —— ①

Diff ① partially w.r.to x

$\frac{\partial z}{\partial y} = p = (1+0)(y+b) = y+b$

Diff ① partially w.r.to y

$\frac{\partial z}{\partial y} = q = (x+a)(1+0) = x+a$

sub p & q in ①

$\therefore z = pq$ is the req PDE.

(iv) Gn: $(x-a)^2+(y-b)^2 = z^2\cot^2\alpha$ —— ①

Diff ① partially w.r.to x

$$2(x-a)+0 = 2z.\frac{\partial z}{\partial x}\cot^2\alpha$$

$$2(x-a) = 2z\frac{\partial z}{\partial x}\cot^2\alpha$$

$$\frac{(x-a)}{\cot^2\alpha} = z.\frac{\partial z}{\partial x} \Rightarrow zp = \frac{(x-a)}{\cot^2\alpha}$$

$$\Rightarrow (x-a) = zp\cot^2\alpha \text{ —— ②}$$

Diff ① partially w.r. to y

$$0+2(y-b) = 2z.\frac{\partial z}{\partial y}\cot^2\alpha$$

$$2(y-b) = 2z.\frac{\partial z}{\partial y}\cot^2\alpha$$

$$y-b = z.\frac{\partial z}{\partial y}\cot^2\alpha$$

$$\Rightarrow y-b = zq\cot^2\alpha \text{ —— ③}$$

sub ② & ③ in ①

$$(zp\cot^2\alpha)^2+(zq\cot^2\alpha)^2 = z^2\cot^2\alpha$$

~~p+q=1p~~

$$z^2p^2\cot^4\alpha + z^2q^2\cot^4\alpha = z^2\cot^2\alpha$$

$$z^2\cot^4\alpha\,(p^2+q^2) = z^2\cot^2\alpha$$

$$p^2+q^2 = \frac{1}{\cot^2\alpha}$$

$$\Rightarrow p^2+q^2 = \tan^2\alpha$$

HW: 1) $z = (x^2+a^2)(y^2+b^2)$ Soln: $pq = 4xyz$

2) $z = a^2x + ay^2 + b$ soln: $4py^2 = q^2$

Gn: $z = a^2x + ay^2 + b$ — ①

Diff ① p.w.r.to. x

$$\frac{\partial z}{\partial x} = p = a^2 + 0 + b = a^2 \text{ — ②}$$

Diff ① p.w.r.to y

$$\frac{\partial z}{\partial y} = q = 0 + 2ay + 0$$

$$q = 2ay$$

$$y = \frac{q}{2a}$$

$$y^2 = \left(\frac{q}{2a}\right)^2 = \frac{q^2}{2a^2}$$

sub ② in the above

$$y^2 = \frac{q^2}{2p} \text{ — ③}$$

sub ② & ③ in ①.

$$z = px + a\left(\frac{q^2}{2p}\right) + b$$

Using ③

$\Rightarrow 2py^2 = q^2$ is the required PDE.

2) Form a pde by eliminating the a.c. a & b from $z = a(x+y) + b$.

Soln: Gn: $z = a(x+y) + b$

Diff ① w.r. to x

$$\frac{\partial z}{\partial x} = p = a$$

Diff ② w.r. to y

$$\frac{\partial z}{\partial y} = q = a$$

$p = a = q \Rightarrow \boxed{p = q}$ is the req. pde.