Problems based on (i) $Z[1] = \frac{Z}{Z-1}$

(ii) $Z[a^n] = \frac{Z}{Z-a}$.

1) Find the Z-transform

(i) $Z(K)$ W.K.T $Z(1) = \frac{Z}{Z-1}$

$$Z(K) = Z(K \cdot 1) = K\,Z(1)$$

$$= K \cdot \frac{Z}{Z-1}.$$

(ii) $Z(-1)^n$ W.K.T $Z[a^n] = \frac{Z}{Z-a}$.

$$Z[(-1)^n] = \frac{Z}{Z-(-1)}$$ Here $a = -1$

$$= \frac{Z}{Z+1}$$

(iii) $Z\left[\frac{1}{7^n}\right]$ W.K.T $Z[a^n] = \frac{Z}{Z-a}$

$$Z\left[\frac{1}{7^n}\right] = Z\left[\left(\frac{1}{7}\right)^n\right] = Z\left[\left(\frac{1}{7}\right)^n\right]$$

Here $a = \frac{1}{7}$.

$$\therefore Z\left[\frac{1}{7^n}\right] = \frac{Z}{Z-\frac{1}{7}} = \frac{7Z}{7Z-1}$$ //

(iv) $Z[e^{an}]$ W.K.T $Z[a^n] = \frac{z}{z-a}$.

$\therefore Z[e^{an}]$ Here $a = e^a$

$$Z[e^{an}] = \frac{z}{z-e^a} \quad //$$

(v) $Z[\cos n\theta]$ and $Z[\sin n\theta]$.

W.K.T

$$Z[a^n] = \frac{z}{z-a}.$$

put $a = e^{i\theta}$

$$Z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}} = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$Z[e^{in\theta}] = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$Z[\cos n\theta + i\sin n\theta] = \frac{z}{z-\cos\theta - i\sin\theta}$$

$$= \frac{z}{(z-\cos\theta)-i\sin\theta} \times \frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$= \frac{z((z-\cos\theta)+i\sin\theta)}{(z-\cos\theta)^2 - i^2\sin^2\theta}.$$

$$= \frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+\cos^2\theta-(-1)\sin^2\theta)}$$

$$= \frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+\cos^2\theta+\sin^2\theta}.$$

$$= \frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+1}$$

$$z[\cos n\theta] + iz[\sin n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Equating real and imaginary parts, we get

$$z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

(ii) $z[r^n\cos n\theta]$ and $z[r^n\sin n\theta]$

we know that $z[a^n] = \frac{z}{z-a}$

put $a = re^{i\theta}$

$$z[(re^{i\theta})^n] = \frac{z}{z-re^{i\theta}}$$

$$z[r^n e^{ni\theta}] = \frac{z}{z-r(\cos\theta + i\sin\theta)}$$

$$z[r^n(\cos n\theta + i\sin\theta)] = \frac{z}{[(z-r\cos\theta)+ir\sin\theta]}$$

$$= \frac{z}{(z-r\cos\theta)+ir\sin\theta} \times \frac{(z-r\cos\theta)-ir\sin\theta}{(z-r\cos\theta)-ir\sin\theta}$$

$$= \frac{z(z-r\cos\theta)-zir\sin\theta}{(z-r\cos\theta)^2-i^2r^2\sin^2\theta}$$

$$= \frac{z(z-r\cos\theta)-zir\sin\theta}{z^2-2zr\cos\theta+r^2\cos^2\theta+r^2\sin^2\theta}$$

$$= \frac{z(z-r\cos\theta)-izr\sin\theta}{z^2-2zr\cos\theta+r^2(\cos^2\theta+\sin^2\theta)}$$

$$= \frac{z(z - r\cos\theta) + irz\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$Z[r^n\cos n\theta + ir^n\sin n\theta] = \frac{z(z - r\cos\theta) + irz\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Equating real and imaginary parts we get,

$$Z[r^n\cos n\theta] = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$$

$$Z[r^n\sin n\theta] = \frac{rz\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Replace r by a to get $Z[a^n\cos n\theta]$ and $Z[a^n\sin n\theta]$.

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1) Find $Z\left[\cos\frac{n\pi}{2}\right]$.

we know that $Z[\cos n\theta] = \dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$

put $\theta = \frac{\pi}{2}$,

$$Z\left[\cos\frac{n\pi}{2}\right] = \frac{z\left(z - \cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1}$$

$$= \frac{z^2}{z^2+1} \quad \left(\text{formula: since } \cos\frac{\pi}{2} = 0\right)$$

2) Find $Z\left[\sin\frac{n\pi}{2}\right]$.

we know that $Z[\sin n\theta] = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

put $\theta = \frac{\pi}{2}$

$$Z\left[\sin\frac{n\pi}{2}\right] = \frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1} = \frac{z}{z^2+1}$$

$\left(\text{formula } \sin\frac{\pi}{2} = 1\right)$