UNIT - V (1)

Z - Transforms and Difference Equations

Z - transform

Let $\{x(n)\}$ be a sequence for $n = \cdots -3, -2, -1, 0, 1, 2, 3, \cdots$ Then the two sided Z - transform of the sequence $x(n)$ is defined as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex number.

One sided Z - transform

Let $\{x(n)\}$ be a sequence for $n = 0, 1, 2, \cdots$ Then the Z - transform of the sequence $\{x(n)\}$ is defined as

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

Definition

If $f(t)$ is a function defined for discrete values of t when $t = nT$, $n = 0, 1, 2, \cdots \infty$ T being the sampling period then the Z-transform of $f(t)$ is defined as

$$Z\{f(t)\} = \sum_{n=0}^{\infty} f(t)\, z^{-n} = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

Properties of Z - transform

1. Linearity

If $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$

then $Z\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 X_1(z) + a_2 X_2(z)$.

Proof

$$Z\left[a_1 x_1(n) + a_2 x_2(n)\right] = \sum_{n=0}^{\infty} \left[a_1 x_1(n) + a_2 x_2(n)\right] z^{-n}$$

$$= a_1 \sum_{n=0}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=0}^{\infty} x_2(n) z^{-n}$$

$$= a_1 X_1(z) + a_2 X_2(z).$$

2. ~~Shifting~~ Damping rule

If $Z\{f(n)\} = F(z)$, then

$$Z\{a^n f(n)\} = F\left(\frac{z}{a}\right)$$ and

$$Z\{a^{-n} f(n)\} = F(az).$$

Proof

$$Z\{a^n f(n)\} = \sum_{n=0}^{\infty} a^n f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n) \left(\frac{z}{a}\right)^{-n} = F\left(\frac{z}{a}\right)$$

$$Z\{a^{-n} f(n)\} = \sum_{n=0}^{\infty} a^{-n} f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n) (az)^{-n}$$

$$= F(az)$$

3. Shifting Theorem

If $Z\{f(n)\} = F(z)$ and $k > 0$, then

$$Z\{f(n-k)\} = \cancel{z}\ z^{k} F(z)$$

and (or) $Z\{f(n+k)\} = z^{k}\left[F(z) - f_0 - f_1 z^{-1} - \cdots - f_{k-1} z^{-(k-1)}\right]$

Proof

$$Z\{f(n-k)\} = \sum_{n=0}^{\infty} f_{n-k} z^{-n}$$

$$= f_0 z^{-k} + f_1 z^{-(k+1)} + \cdots$$

$$= z^{-k}\left[f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots\right]$$

$$= z^{-k} \sum_{n=0}^{\infty} f_n z^{-n}$$

$$= z^{-k} F(z).$$

$$Z\{f_{n+k}\} = \sum_{n=0}^{\infty} f_{n+k}\, z^{-n}$$

$$= z^{k} \sum_{n=0}^{\infty} f_{n+k}\, z^{-(n+k)}$$

$$= z^{k}\left[f_k z^{-k} + f_{k+1} z^{-(k+1)} + \cdots\right]$$

$$= z^{k}\left[\sum_{n=0}^{\infty} f_n z^{-n} - \sum_{n=0}^{k-1} f_n z^{-n}\right]$$

$$= z^{k}\left[F(z) - f_0 - f_1 z^{-1} - f_2 z^{-2} - \cdots - f_{k-1} z^{-(k-1)}\right]$$

Initial Value Theorem

If $Z\{f(n)\} = F(z)$, then $f(0) = \lim_{z\to\infty} F(z)$

Proof :

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$F(z) = f(0) + f(1) z^{-1} + f(2) z^{-2} + \cdots$$

$$\boxed{\lim_{n\to\infty} F(z) = f(0)}$$