2) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, then find the displacement.

Soln.

The ODWE is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

i). $y(0,t) = 0, \quad \forall t$

ii). $y(l,t) = 0, \quad \forall t$

iii). $\frac{\partial}{\partial t} y(x,0) = 0, \quad \forall x$

iv). $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$

The suitable solution is,

$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \rightarrow (1)$

Applying (i) in (1), we get

$y(0,t) = 0$

$(A(1) + B(0))(C \cos pat + D \sin pat) = 0$

$A(C \cos pat + D \sin pat) = 0$

$C \cos pat + D \sin pat \neq 0$ (It is a fn. of 't')

$\Rightarrow \boxed{A=0}$

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Subs. $A=0$ in (1),

$$y(x,t) = B \sin px \,(C \cos pat + D \sin pat) \rightarrow (2)$$

Applying ii) in (2),

$$y(l,t)=0$$

$$B \sin pl \,(C \cos pat + D \sin pat) = 0$$

$B \neq 0$ (If $B=0$, we get a trivial Soln.)

$C \cos pat + D \sin pat \neq 0$ ($\because$ it is a fn. of 't')

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Subs. $p = \frac{n\pi}{l}$ in (2),

$$y(x,t) = B \sin \frac{n\pi x}{l}\left[C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l}\right] \rightarrow (3)$$

Before applying iii)., differentiate partially (3) w.r.t 't'

$$\frac{\partial}{\partial t} y(x,t) = B \sin \frac{n\pi x}{l}\left[-C \sin \frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right) + D \cos \frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\right]$$

Applying condition iii)., we get

$$\frac{\partial}{\partial t} y(x,0) = 0$$

$$B \sin \frac{n\pi x}{l}\left[0 + \frac{n\pi a}{l} D\right] = 0$$

$$B D \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = 0$$

Here $B \neq 0$ [If $B=0$, we get a trivial soln.]

$\sin \frac{n\pi x}{l} \neq 0$ [$\because$ it is a fn. of x]

$\Rightarrow \boxed{D=0}$

Subs. $D=0$ in (3),

$$y(x,t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi a t}{l} \right)$$

$$= BC \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

$$y(x,t) = B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

The most general soln. is,

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \rightarrow (4)$$

Applying condition iv) in (4), we get

$$y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{y_0}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right]$$

$$\Rightarrow B_1 \sin \frac{\pi x}{l} + B_2 \sin \frac{2\pi x}{l} + B_3 \sin \frac{3\pi x}{l} + \dots$$

$$= \frac{3y_0}{4} \sin \frac{\pi x}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l}$$

Equating the like coefficients, we get

$$B_1 = \frac{3y_0}{4} \; ; \quad B_2 = 0 \; ; \quad B_3 = -\frac{y_0}{4} \; ; \quad B_4 = B_5 = \dots = 0$$

Subs the above values in (4),

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l}$$

Hw I. displacing position $y(x,0) = k \sin \frac{3\pi x}{l} \cos \frac{2\pi x}{l}$

2. a sinusoidal arc of height y_0

$\hookrightarrow y_0 \sin \frac{\pi x}{l}$

Type-II

Problems based on vibrating string with non-zero initial velocity:

I. A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating string giving each point a velocity $\lambda x(l-x)$. Find the ~~Show that~~ displacement.

Soln. :

The ODWE is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are,

i). $y(0,t)=0, \quad \forall t$

ii). $y(l,t)=0, \quad \forall t$

iii). $y(x,0)=0, \quad \forall x$

iv). $\frac{\partial}{\partial t} y(x,0) = \lambda x(l-x)$

The suitable solution is,

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \rightarrow (1)$$

Applying condn. (i) in (1), we get

$$y(0,t)=0$$

$$(A(1)+B(0))(C\cos pat + D\sin pat) = 0$$

$$A(C\cos pat + D\sin pat) = 0 \quad [\because \text{it is a fn. of 't'}]$$

Here $C\cos pat + D\sin pat \neq 0$

$$\boxed{A=0}$$

$(1) \Rightarrow y(x,t) = B\sin px\,(C\cos pat + D\sin pat) \rightarrow (2)$

Applying condn. (ii) in (2), we get

$$y(l, t) = 0$$

$$B \sin pl \,(C \cos pat + D \sin pat) = 0$$

Here $C \cos pat + D \sin pat \neq 0$ ($\because$ It is a fn. of 't')

and $B \neq 0$ (If $B = 0$, then we get a trivial soln.)

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

$$(2) \Rightarrow y(x, t) = B \sin \frac{n\pi x}{l}\left(C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l}\right) \rightarrow (3)$$

Applying condn. (iii) in (3),

$$y(x, 0) = 0$$

$$B \sin \frac{n\pi x}{l}\left(C(1) + D(0)\right) = 0$$

$$BC \sin \frac{n\pi x}{l} = 0$$

Here $B \neq 0$ (Already explained)

$$\sin \frac{n\pi x}{l} \neq 0$$

$$\boxed{C = 0}$$

$$(3) \Rightarrow y(x, t) = B \sin \frac{n\pi x}{l}\left[0 + D \sin \frac{n\pi at}{l}\right]$$

$$= BD \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$$

$$= B \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \quad \text{where } B = BD$$

The most general soln. is,

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \rightarrow (4)$$

Before applying condn. (iv), differentiate (4) partially w.r.t. 't', we get

$$\frac{\partial}{\partial t} y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l} \left(\frac{n\pi a}{l}\right)$$

Applying condn. iv)

$$\frac{\partial}{\partial t} y(x,0) = \lambda x(l-x)$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\left(\frac{n\pi a}{l}\right) = \lambda(lx-x^2)$$

HRSS $\displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \lambda(lx-x^2)$ where $b_n = B_n\left(\frac{n\pi a}{l}\right)$

To find b_n:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \lambda(lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\frac{-l}{n\pi}(lx-x^2)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}(l-2x)\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\left(0-0-\frac{2l^3}{n^3\pi^3}(-1)^n\right)-\left(0-0-\frac{2l^3}{n^3\pi^3}\right)\right]$$

$$= \frac{2\lambda}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n+\frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2\lambda}{l}\,\frac{2l^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n\frac{n\pi a}{l} = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n = \frac{l}{n\pi a}\,\frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{4\lambda l^3}{n^4\pi^4 a}\left[1-(-1)^n\right]$$

$$B_n = \begin{cases} \dfrac{8\lambda l^3}{n^4\pi^4 a}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$(5) \Rightarrow y(x,t) = \sum_{n=odd}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l}$$

$$= \sum_{n=odd}^{\infty}\frac{8\lambda l^3}{n^4\pi^4 a}\sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l}$$

$$= \frac{8\lambda l^3}{\pi^4 a}\sum_{n=odd}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l}$$