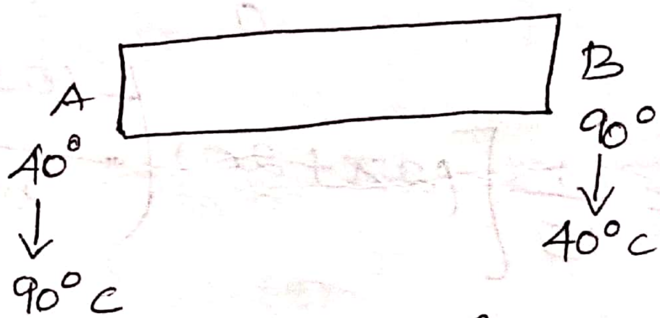
Type 2: Steady state conditions with non zero boundary conditions

1. The ends A and B of a rod of length of 'l' have the temp. $40^\circ c$ and $90^\circ c$ until steady state prevails. The temp. at A is suddenly raised to $90^\circ c$ and at the same time that at B is lowered to $40^\circ c$. Find the temperature distribution in the rod at time 't'.

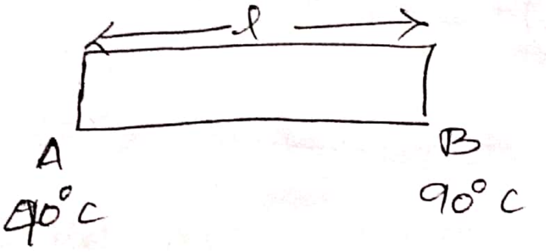
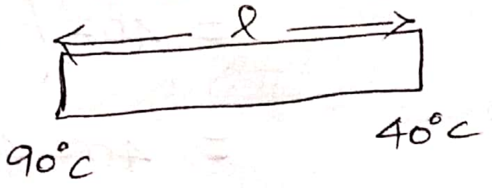
Soln.:



The one dimensional Heat eqn. is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

First we find the temperature function $u(x,t)$ at any distance before and after the changes of temperature at the ends of the rod. So there are two steady state solns.

Before change	After change
	
$u(x) = \left(\frac{b-a}{l}\right)x + a$	$f(x) = \left(\frac{b-a}{l}\right)x + a$
$= \left(\frac{90-40}{l}\right)x + 40$	$= \left(\frac{40-90}{l}\right)x + 90$
$= \frac{50x}{l} + 40$	$= -\frac{50}{l}x + 90$

The initial temperature is

$$u(x,0) = \frac{50x}{l} + 40$$

The boundary conditions are

i). $u(0,t) = 90$

ii). $u(l,t) = 40$

iii). $u(x,0) = \frac{50x}{l} + 40$

Since the boundary conditions are non-zero,

$\therefore$ we assume that

$$u(x,t) = f(x) + V(x,t) \rightarrow (1)$$

$$\Rightarrow V(x,t) = u(x,t) - f(x)$$

a). $V(0,t) = u(0,t) - f(0)$

$= 90 - 90$

$= 0$

b). $V(l,t) = u(l,t) - f(l)$

$= 40 - \left(-\frac{50l}{l} + 90\right)$

$= 40 - 40$

$= 0$

c). $V(x,0) = u(x,0) - f(x)$

$= \frac{50x}{l} + 40 - \left(-\frac{50x}{l} + 90\right)$

$= \frac{100x}{l} - 50$

The new boundary conditions are

a). $V(0,t) = 0$

b). $V(l,t) = 0$

c). $V(x,0) = \frac{100x}{l} - 50$

The suitable soln. is,

$$V(x,t) = (A\cos px + B\sin px)\, e^{-a^2p^2t} \rightarrow (2)$$

(a) $\Rightarrow$ $V(0,t) = 0$

$A\, e^{-a^2p^2t} = 0$

Here $e^{-a^2p^2t} \neq 0$ [$\because$ It is a fn. of 't']

$\Rightarrow A = 0$

(2) $\Rightarrow$ $V(x,t) = B\sin px\, e^{-a^2p^2t} \longrightarrow (3)$

(b) $\Rightarrow$ $V(l, t) = 0$

$$B \sin pl \; e^{-a^2 p^2 t} = 0$$

Here $e^{-a^2 p^2 t} \neq 0$

$B \neq 0$ [Suppose $B=0$, we get a trivial soln.]

$\Rightarrow \sin pl = 0$

$$pl = n\pi \Rightarrow p = \frac{n\pi}{l}$$

$$(3) \Rightarrow V(x,t) = B \sin \frac{n\pi x}{l} \, e^{-a^2 \frac{n^2\pi^2}{l^2} t} \rightarrow (4)$$

The most general soln. is,

$$V(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \, e^{-\frac{a^2 n^2 \pi^2}{l^2} t} \rightarrow (5)$$

(c) $\Rightarrow$ $V(x,0) = \frac{100x}{l} - 50$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \, e^{-a^2 \frac{n^2\pi^2}{l^2}(0)} = \frac{100x}{l} - 50$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{100x}{l} - 50$$

HRSS $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = \frac{100x}{l} - 50$

$\therefore b_n = B_n$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$B_n = \frac{2}{l}\int_0^l \left(\frac{100x}{l} - 50\right) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l}\left[\left(\frac{100x}{l} - 50\right)\left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l}\right) - \left(\frac{100}{l}\right)\left(\frac{-\sin \frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) + 0\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l}{n\pi}\left(\frac{100x}{l} - 50\right)\cos \frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\left(\frac{100}{l}\right)\sin \frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{l}\left[\left(\frac{-l}{n\pi} 50(-1)^n + 0\right) - \left(\frac{-l}{n\pi}(-50) + 0\right)\right]$$

$$= \frac{2}{l}\left[\frac{-50l}{n\pi}(-1)^n - \frac{50l}{n\pi}\right]$$

$$= \frac{-2}{l}\frac{50l}{n\pi}\left[1+(-1)^n\right] = \frac{-100}{n\pi}\left[1+(-1)^n\right]$$

$$B_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{-200}{n\pi}, & \text{if } n \text{ is even} \end{cases}$$

$$(5) \Rightarrow v(x,t) = \sum_{n=\text{even}}^{\infty} -\frac{200}{n\pi} \sin\frac{n\pi x}{l}\, e^{\frac{-a^2 n^2 \pi^2 t}{l^2}}$$

$$= -\frac{200}{\pi}\sum_{n=\text{even}}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{l}\, e^{\frac{-a^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = -\frac{50x}{l} + 90 - \frac{200}{\pi}\sum_{n=\text{even}}^{\infty} \frac{1}{n}\sin\frac{n\pi x}{l}\, e^{\frac{-a^2 n^2 \pi^2 t}{l^2}}$$

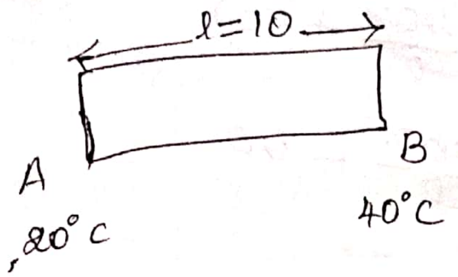
2]. A bar 10 cm long with insulated sides, has its ends A and B kept at 20°c and 40°c resly. until steady state condns. prevail. The temperature at A is then suddenly raised to 50°c and at the same instant that at B is lowered to 10°c. Find the subsequent temperature at any pt. of the bar at any time.

Soln.:

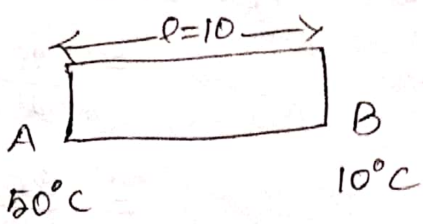
The ODHE is $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$

First we find the temperature function $u(x,t)$ at any distance before and after the changes of temperature at the ends of the rod. So there are two steady state solutions.

Before change



After change



$$u(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{40-20}{l}\right)x + 20$$

$$= \frac{20x}{l} + 20$$

$$f(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{10-50}{l}\right)x + 50$$

$$= -\frac{40x}{l} + 50$$

The initial temperature is

$$u(x,0) = \frac{20x}{l} + 20$$

The boundary conditions are,

i). $u(0,t) = 50$

ii). $u(l,t) = 10$

iii). $u(x,0) = \frac{20x}{l} + 20$

Since the boundary conditions are non zero.

$\therefore$ we assume that

$$u(x,t) = f(x) + v(x,t) \rightarrow (1)$$

$$\Rightarrow v(x,t) = u(x,t) - f(x)$$

a). $v(0,t) = u(0,t) - f(0)$

$$= 50 - 50 = 0$$

b). $v(l,t) = u(l,t) - f(l)$

$$= 10 - 10 = 0$$

c). $v(x,0) = u(x,0) - f(x)$

$$= \frac{20x}{l} + 20 - \left(-\frac{40x}{l} + 50\right)$$

$$= \frac{20x}{l} + 20 + \frac{40x}{l} - 50$$

$$= \frac{60x}{l} - 30$$

The new boundary conditions are,

a). $v(0,t) = 0$

b). $v(l,t) = 0$

c). $v(x,0) = \frac{60x}{l} - 30$

The suitable soln: is,

$$v(x,t) = (A\cos px + B\sin px)\, e^{-a^2p^2t} \rightarrow (2)$$

(a) $\Rightarrow$ $v(0,t) = 0$

$$A\, e^{-a^2p^2t} = 0$$

Here $e^{-a^2p^2t} \neq 0$ [It is a fn. of 't']

$\Rightarrow A = 0$

$$(2) \Rightarrow v(x,t) = B\sin px\, e^{-a^2p^2t} \rightarrow (3)$$

(b) $\Rightarrow$ $v(l,t) = 0$

$$B\sin pl\, e^{-a^2p^2t} = 0$$

Here $B \neq 0$ (Suppose $B=0$, we get a trivial soln.)

and $e^{-a^2p^2t} \neq 0$ (It is a fn. of 't')

$\Rightarrow \sin pl = 0$

$pl = n\pi \Rightarrow p = \frac{n\pi}{l}$

$(3) \Rightarrow \quad v(x,t) = B \sin \frac{n\pi x}{l} \; e^{-a^2 \frac{n^2\pi^2}{l^2} t} \longrightarrow (4)$

The most general soln. is,

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \; e^{-a^2 \frac{n^2\pi^2}{l^2} t} \longrightarrow (5)$$

$(c) \Rightarrow \quad v(x,0) = \frac{60x}{l} - 30$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{60x}{l} - 30$$

HRSS $\quad \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = \frac{60x}{l} - 30 \quad [B_n = b_n]$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$B_n = \frac{2}{l}\int_0^l \left(\frac{60x}{l} - 30\right) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\left(\frac{60x}{l} - 30\right)\left(\frac{-\cos \frac{n\pi x}{l}}{n\pi/l}\right) - \left(\frac{60}{l}\right)\left(-\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 0\right]_0^l$$

$$= \frac{2}{l}\left[\left(\frac{-l}{n\pi}(30)(-1)^n + \frac{60}{l}\frac{l^2}{n^2\pi^2}(0)\right) - \left(30\frac{l}{n\pi} + 0\right)\right]$$

$$= \frac{2}{l}\left[-\frac{30l}{n\pi}(-1)^n - \frac{30l}{n\pi}\right]$$

$$= \frac{-60l}{l n\pi}\left[1 + (-1)^n\right]$$

$$= \frac{-60}{n\pi}\left[1 + (-1)^n\right]$$

$$B_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{-120}{n\pi}, & \text{if } n \text{ is even} \end{cases}$$

$$(5) \Rightarrow V(x,t) = \sum_{n=\text{even}}^{\infty} \frac{-120}{n\pi} \sin \frac{n\pi x}{l}\, e^{-\frac{a^2 n^2 \pi^2}{l^2} t}$$

$$\therefore \quad u(x,t) = f(x) + v(x,t)$$

$$u(x,t) = -\frac{40x}{l} + 50 - \frac{120}{\pi} \sum_{n=\text{even}}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l}\, e^{-\frac{a^2 n^2 \pi^2}{l^2} t}$$

Since $l = 10$ cm,

$$u(x,t) = -\frac{40x}{10} + 50 - \frac{120}{\pi} \sum_{n=\text{even}}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{10}\, e^{-\frac{a^2 n^2 \pi^2 t}{100}}$$

$$= -4x + 50 - \frac{120}{\pi} \sum_{n=\text{even}}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{10}\, e^{-\frac{a^2 n^2 \pi^2 t}{100}}$$