Introduction :

The differential Equation for two dimensional heat flow for the unsteady case is.

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

In steady state, $u$ is independent of $t$ i.e., $\frac{\partial u}{\partial t} = 0$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{[Laplace eqn.]}$$

possible solns. :

i). $u(x,y) = (A_1 e^{px} + A_2 e^{-px})(A_3 \cos py + A_4 \sin py)$

ii). $u(x,y) = (A_5 \cos px + A_6 \sin px)(A_7 e^{py} + A_8 e^{-py})$

iii). $u(x,y) = (A_9 x + A_{10})(A_{11} y + A_{12})$

~~Suitable soln.~~

i). If heat flows in $x$- direction, then

$$u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$$

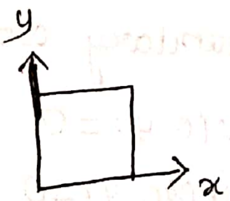
ii). If heat flows in $y$- direction, then

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$$

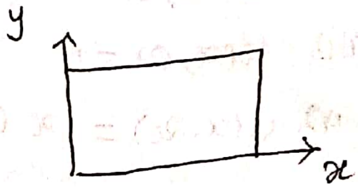
Types of plates :

I. Finite plates

↳ i). Square plate

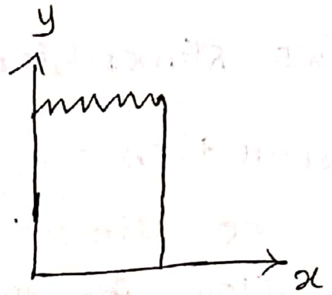


ii). Rectangular plate

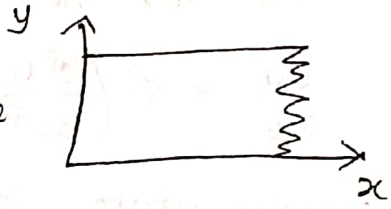


2]. Infinite plates

i). Vertically Infinite plate




ii). Horizondally Infinite plate



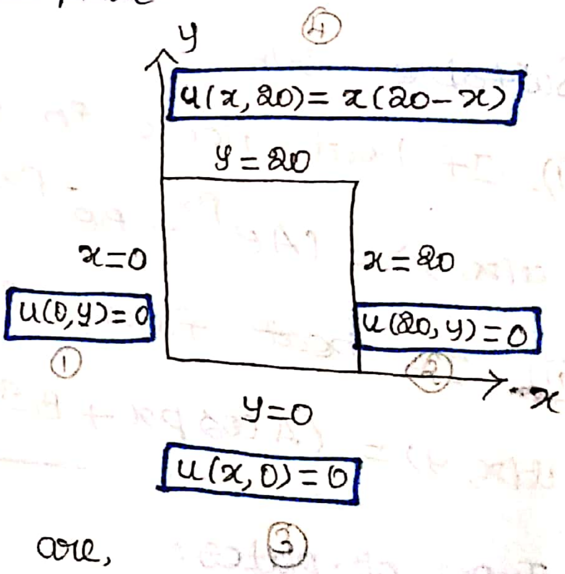

Finite plates

1]. The square plate bounded by the line $x=0, y=0, x=20$ and $y=20$. It's faces are insulated. The temperature along the upper horizontal edge is given by $u(x,20) = x(20-x)$ while the other three edges are kept at $0°c$. Find the steady State temperature in the plate.

Soln.:

The laplace eqn. is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$




The boundary conditions are,

i). $u(0,y)=0$

ii). $u(20,y)=0$

iii). $u(x,0)=0$

iv). $u(x,20) = x(20-x)$



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The suitable soln. is,

$$u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \rightarrow (1)$$

$(i) \Rightarrow \quad u(0, y) = 0$

$$(A(1) + B(0))(Ce^{py} + De^{-py}) = 0$$

$$A(Ce^{py} + De^{-py}) = 0$$

Here $Ce^{py} + De^{-py} \neq 0$ ($\because$ it is a fn. of '$y$')

$$\Rightarrow A = 0$$

$(1) \Rightarrow \quad u(x, y) = B\sin px\,(Ce^{py} + De^{-py}) \rightarrow (2)$

$(ii) \Rightarrow \quad u(20, y) = 0$

$$B\sin 20p\,(Ce^{py} + De^{-py}) = 0$$

Here $Ce^{py} + De^{-py} \neq 0$

and $B \neq 0$

$$\Rightarrow \sin 20p = 0$$

$$20p = n\pi$$

$$p = \frac{n\pi}{20}$$

$(2) \Rightarrow \quad u(x, y) = B\sin\frac{n\pi x}{20}\left(Ce^{\frac{n\pi y}{20}} + De^{\frac{-n\pi y}{20}}\right) \rightarrow (3)$

$(iii) \Rightarrow \quad u(x, 0) = 0$

$$B\sin\frac{n\pi x}{20}[C + D] = 0$$

Here $B \neq 0$

$$\sin\frac{n\pi x}{20} \neq 0$$

$$\Rightarrow C + D = 0 \Rightarrow D = -C$$



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$$(3) \Rightarrow u(x,y) = B \sin\frac{n\pi x}{20}\left[c\, e^{\frac{n\pi y}{20}} - c\, e^{-\frac{n\pi y}{20}}\right]$$

$$= Bc \sin\frac{n\pi x}{20}\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]$$

$$= 2Bc \sin\frac{n\pi x}{20} \sinh\frac{n\pi y}{20}$$

$$= B \sin\frac{n\pi x}{20} \sinh\frac{n\pi y}{20}$$

$$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

$$e^{\theta} - e^{-\theta} = 2\sinh\theta$$

where $2BC = B$

The most general soln. is,

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{20} \sinh\frac{n\pi y}{20} \rightarrow (4)$$

(iv) $\Rightarrow$ $u(x, 20) = x(20-x)$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{20} \sinh n\pi = x(20-x)$$

$$\sum_{n=1}^{\infty} \left[B_n \sinh n\pi\right] \sin\frac{n\pi x}{20} = x(20-x)$$

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$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20} = x(20-x)$$

$$\Rightarrow b_n = B_n \sinh n\pi$$

Now, $$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{20}\int_0^{20} (20x - x^2) \sin\frac{n\pi x}{20}\, dx$$

$$= \frac{1}{10}\int_0^{20} \underbrace{(20x - x^2)}_{u} \underbrace{\sin\frac{n\pi x}{20}}_{v}\, dx$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$



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$$= \frac{1}{10}\left[(20x - x^2)\left(\frac{\cos\frac{n\pi x}{20}}{n\pi/20}\right) - (20-2x)\left(\frac{\sin\frac{n\pi x}{20}}{n^2\pi^2/400}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{20}}{n^3\pi^3/8000}\right) - 0\right]_0^{20}$$

$$= \frac{1}{10}\left[\left(\boxed{-\frac{20}{n\pi}(10)} + 0 - \frac{16000}{n^3\pi^3}(-1)^n\right) - \left(0 + 0 - \frac{16000}{n^3\pi^3}\right)\right]$$

$$\boxed{=\frac{1}{10}}\left(-\frac{16000}{n^3\pi^3}(-1)^n + \frac{16000}{n^3\pi^3}\right)$$

$$b_n = \frac{16000}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$B_n \sinh n\pi = \frac{16000}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n = \frac{16000}{n^3\pi^3 \sinh n\pi}\left[1-(-1)^n\right]$$

$$B_n = \begin{cases} \frac{32000}{n^3\pi^3 \sinh n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$(4) \Rightarrow u(x,y) = \sum_{n=odd}^{\infty} \frac{3200}{n^3\pi^3 \sinh n\pi} \sin\frac{n\pi x}{20} \sinh\frac{n\pi y}{20}$$

$$u(x,y) = \frac{3200}{\pi^3}\sum_{n=odd}^{\infty} \frac{1}{n^3 \sinh n\pi} \sin\frac{n\pi x}{20} \sinh\frac{n\pi y}{20}$$



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## Infinite plate:

I. An infinitely long rectangular plate with insulated surface is 10cm wide. The two long edges and one short edge are kept at zero temperature. while the other short edge $x=0$ is kept at temp. given by,

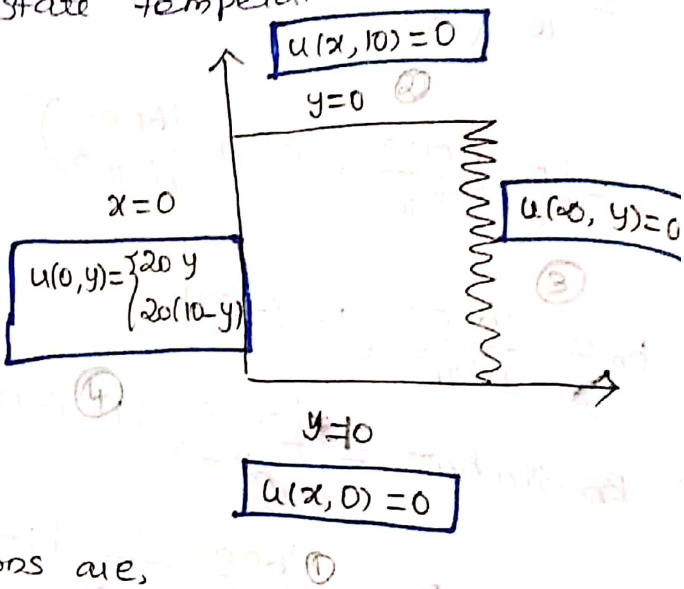
$$u = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases}$$

Find the steady state temperature distribution in the plate.

Soln.

The laplace eqn. is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$




The boundary conditions are,

i). $u(x,0)=0$

ii). $u(x,10)=0$

iii). $u(\infty,y)=0$

iv). $u(0,y) = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases}$

The suitable Soln. is,

$$u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \rightarrow (1)$$

(i) $\Rightarrow u(x,0)=0$

$$(Ae^{px} + Be^{-px})\,C = 0$$

Here $Ae^{px} + Be^{-px} \neq 0$ [$\because$ it is a fn. of $x$]

$$\Rightarrow \quad C = 0$$



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$(1) \Rightarrow u(x, y) = (Ae^{px} + Be^{-px}) D \sin py \rightarrow (2)$

(ii) $\Rightarrow$ $u(x, 10) = 0$

$(Ae^{px} + Be^{-px}) D \sin 10p = 0$

Here $Ae^{px} + Be^{-px} \neq 0$

$D \neq 0$ [If $D=0$, then we get a trivial Soln.]

$\Rightarrow \sin 10p = 0$

$10p = n\pi \Rightarrow p = \frac{n\pi}{10}$

$(2) \Rightarrow u(x, y) = (Ae^{n\pi x/10} + Be^{-n\pi x/10}) D \sin \frac{n\pi y}{10} \rightarrow (3)$

(iii) $\Rightarrow$ $u(\infty, y) = 0$

$(Ae^{\infty} + Be^{-\infty}) D \sin \frac{n\pi y}{10} = 0$

$Ae^{\infty} D \sin \frac{n\pi y}{10} = 0$

Here $e^{\infty} \neq 0$ and $\sin \frac{n\pi y}{10} \neq 0$

$D \neq 0$

$\Rightarrow A = 0$

$(3) \Rightarrow u(x, y) = Be^{-n\pi x/10} D \sin \frac{n\pi y}{10}$ where $BD = B$

$= Be^{-n\pi x/10} \sin \frac{n\pi y}{10}$

The most general soln is,

$u(x, y) = \sum_{n=1}^{\infty} B_n e^{-n\pi x/10} \sin \frac{n\pi y}{10} \rightarrow (4)$

(iv) $\Rightarrow$ $u(0, y) = \begin{cases} 20y, & 0 \leq y \leq 5 \\ 20(10-y), & 5 \leq y \leq 10 \end{cases}$

$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi y}{10} = \begin{cases} 20y, & 0 \leq y \leq 5 \\ 20(10-y), & 5 \leq y \leq 10 \end{cases}$

Half range Sine Series:

$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{10} = \begin{cases} 20y, & 0 \leq y \leq 5 \\ 20(10-y), & 5 \leq y \leq 10 \end{cases}$

$\therefore b_n = B_n$



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Now,

$$b_n = \frac{2}{10}\int_0^{10} f(y)\,\sin\frac{n\pi y}{10}\,dy$$

$$B_n = \frac{1}{5}\left[\int_0^5 20y\,\sin\frac{n\pi y}{10}\,dy + \int_5^{10} 20(10-y)\sin\frac{n\pi y}{10}\,dy\right]$$

$$= \frac{20}{5}\left[\int_0^5 y\sin\frac{n\pi y}{10}\,dy + \int_5^{10}(10-y)\sin\frac{n\pi y}{10}\,dy\right]$$

$$= 4\left[\left\{y\left(\frac{-\cos\frac{n\pi y}{10}}{n\pi/10}\right) - 1\left(\frac{-\sin\frac{n\pi y}{10}}{n^2\pi^2/100}\right)\right\}_0^5 + \left\{(10-y)\left(\frac{-\cos\frac{n\pi y}{10}}{n\pi/10}\right) - (-1)\left(\frac{-\sin\frac{n\pi y}{10}}{n^2\pi^2/100}\right)\right\}_5^{10}\right]$$

$$= 4\left[\left(-\frac{10y}{n\pi}\cos\frac{n\pi y}{10} + \frac{100}{n^2\pi^2}\sin\frac{n\pi y}{10}\right)_0^5 + \left(-\frac{10}{n\pi}(10-y)\cos\frac{n\pi y}{10} - \frac{100}{n^2\pi^2}\sin\frac{n\pi y}{10}\right)_5^{10}\right]$$

$$= 4\left[\left(-\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + 0\right) + \left(0 - \left(-\frac{50}{n\pi}\cos\frac{n\pi}{2} - \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right)\right]$$

$$= 4\left[\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$B_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \begin{cases} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}; & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$(4) \Rightarrow u(x,y) = \sum_{n=odd} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\, e^{-n\pi x/10}\,\sin\frac{n\pi y}{10}$$



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2). A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered as infinite without introducing the error. The temperature at short edge $y=0$ is gvn. by,

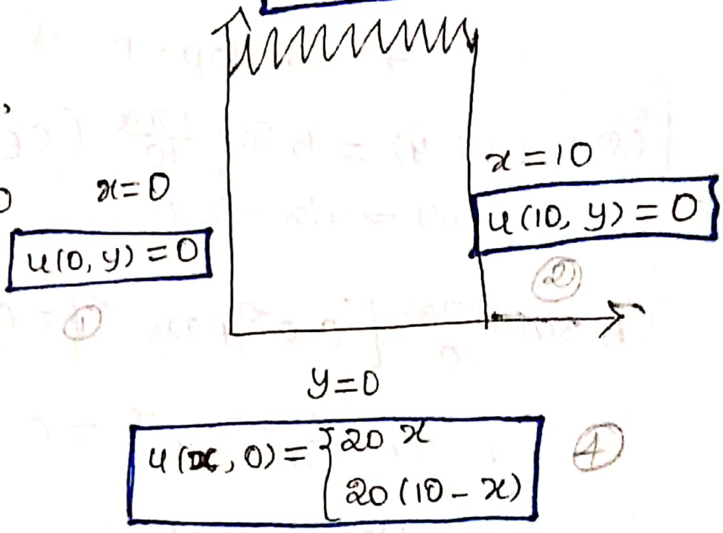
$$u = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 \le x \le 10 \end{cases}$$

and the two long edges $x=0$ and $x=10$ as well as the other short edges ~~x=0~~ are kept at $0^\circ$ c. Find $u(x,y)$.

Soln.

The laplace eqn. is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$




The boundary conditions are,

i). $u(0,y) = 0$

ii). $u(10,y) = 0$

iii) $u(x,\infty) = 0$

iv). $u(x,0) = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 \le x \le 10 \end{cases}$

The suitable soln. is,

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \rightarrow (1)$$

i) ⇒ $u(0,y) = 0$

$$A(Ce^{py} + De^{-py}) = 0$$

Here $ce^{py} + De^{-py} \neq 0$

$\Rightarrow A = 0$

$(1) \Rightarrow u(x, y) = B \sin px \, (ce^{py} + De^{-py}) \rightarrow (2)$

ii) $\Rightarrow u(10, y) = 0$

$B \sin p10 \, (ce^{py} + De^{-py}) = 0$

Here $B \neq 0$

and $\boxed{ce^{py} +} De^{-py} \neq 0$

$\Rightarrow \sin 10p = 0 \Rightarrow p = \frac{n\pi}{10}$

$(2) \Rightarrow u(x, y) = B \sin \frac{n\pi x}{10} \left(ce^{\frac{n\pi y}{10}} + De^{-\frac{n\pi y}{10}}\right) \rightarrow (3)$

iii). $\Rightarrow u(x, \infty) = 0$

$B \sin \frac{n\pi x}{10} \left[ce^{\infty} + De^{-\infty}\right] = 0$

$\boxed{B \sin \frac{n\pi x}{10} \, ce^{\infty}} = 0$

Here $B \neq 0$

$\sin \frac{n\pi x}{10} \neq 0$

$e^{\infty} \neq 0$

$\Rightarrow c = 0$

$(3) \Rightarrow u(x, y) = B \sin \frac{n\pi x}{10} De^{-\frac{n\pi y}{10}}$

$= B \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}}$

The most general soln. is,

$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} e^{-n\pi y/10} \longrightarrow (4)$

iv) $\Rightarrow u(x, 0) = \begin{cases} 20x, & 0 \leq x \leq 5 \\ 20(10-x), & 5 \leq x \leq 10 \end{cases}$

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$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 \le x \le 10 \end{cases}$$

Half Range Sine Series:

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 \le x \le 10 \end{cases}$$

$\therefore\ b_n = B_n$

now $\quad b_n = \dfrac{2}{10}\displaystyle\int_0^{10} f(x) \sin\frac{n\pi x}{10}\, dx$

$$= \frac{1}{5}\left[\int_0^{5} 20x \sin\frac{n\pi x}{10}\, dx + \int_5^{10} 20(10-x) \sin\frac{n\pi x}{10}\, dx\right]$$

$$= 4\left[\int_0^{5} x \sin\frac{n\pi x}{10}\, dx + \int_5^{10} (10-x) \sin\frac{n\pi x}{10}\, dx\right]$$

$$= 4\left[\left(x\,\frac{-\cos\frac{n\pi x}{10}}{n\pi/10} - 1\left(-\frac{\sin\frac{n\pi x}{10}}{n^2\pi^2/100}\right)\right)_0^5 + \left((10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{n\pi/10}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{10}}{n^2\pi^2/100}\right)\right)_5^{10}\right]$$

$$= 4\left[\left(\frac{-10x}{n\pi}\cos\frac{n\pi x}{10} + \frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right)_0^5 + \left(\frac{-10}{n\pi}(10-x)\cos\frac{n\pi x}{10} - \frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right)_5^{10}\right]$$

$$= 4\left[\left(\frac{-50x}{n\pi}\cos\frac{n\pi}{2} + \frac{50}{n\pi}(1)\right) + \left(0 - \left(\frac{-50}{n\pi}\cos\frac{n\pi}{2}\right.\right.\right.$$

$$= 4\left[\left(-\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right) -\right.$$

$$\left.\left(-\frac{50}{n\pi}\cos\frac{n\pi}{2} - \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= 4\left[\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$B_n = \begin{cases} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$(4) \Rightarrow u(x, y) = \sum_{n=\text{odd}}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}$$

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HW J. A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in the length without introducing an appreciable error. If the temp. along one short edge $y=0$ is gvn. by $u(x,0) = 100\sin\frac{\pi x}{8}$ in $0<x<8$ while the two long edges $x=0$ and $x=8$ as well as the other short edges are kept at $0°c$, find the steady state temp. fn. $u(x, y)$.

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