Reduce the quadratic form $2x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 = 0$ to canonical form by orthogonal reduction .Find rank, index, signature and nature

Step 1:

The matrix form is

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2:

Characteristic equation ,Eigen values, Eigen vectors

C_1 =Sum of leading diagonal elements

=2+2+1 =5

C_2= Sum of minors of leading diagonal elements

=4

$C_3=|A|$

$$= \begin{vmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

= 0

The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 4\lambda = 0$$

The eigen values are 0,1,4

The eigen vectors are $(A - \lambda I)X$=0

$$\left[\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

CASE (i)

When $\lambda = 0$

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The cofactor of first row elements are $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$ ie $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

The Eigen vector when $\lambda = 0$ is $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

CASE (ii)

When $\lambda = 1$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The cofactor of third row elements are $\begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}$ ie $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

The Eigen vector when $\lambda = 1$ is $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

CASE (iii)

When $\lambda = 4$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The cofactor of first row elements are $\begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix}$ ie $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

The Eigen vector when $\lambda = 4$ is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

STEP 3:

To check pair wise orthogonality

$$X_1^T X_2 = \begin{pmatrix} 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$X_2^T X_3 = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$X_3^T X_1 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = 0$$

STEP 4:

To find normalized vector

Eigen vector	$l(x)=\sqrt{x_1^2+x_2^2+x_3^2}$	Normalized vector $=\begin{pmatrix} x_1/l(x_1) \\ x_2/l(x_2) \\ x_3/l(x_3) \end{pmatrix}$
$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$	$\sqrt{1+1+0}=\sqrt{2}$	$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$	$\sqrt{0+0+1}=\sqrt{1}$	$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\sqrt{1+1+0}=\sqrt{2}$	$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

STEP 5:

Normalized modal matrix

$$N=\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{bmatrix}$$

$$N^T=\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix}$$

STEP 6:

$NN^T = N^TN = I$

$$N^TN = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I$$

STEP 7:

To find diagonalyze matrix

$$N^TAN = D$$

$$N^TA = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 4/\sqrt{2} & 4/\sqrt{2} & 0 \end{pmatrix}$$

$$N^TAN = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 4/\sqrt{2} & 4/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= D$$

Step 8:

$Y^TDY = 0$

$0y_1^2 + y_2^2 + 4y_3^2 = 0$

The index p=2

Rank r=2

Signature s=2p-r =2

The nature is semi positive