Definition

Let $f(x,y,\alpha)=0$ be a single parameter family of curves, where α is the parameter. The envelope of this family of curves is a curve which touches every member of the family.

1. Find the envelope of the family of straight lines $y=mx+\frac{1}{m}$.

Soln: $y=mx+\frac{1}{m}\Rightarrow my=m^2x+1$

$\Rightarrow xm^2-ym+1=0,$ which is a quadratic in the parameter *m*.

Here $A=x,\ B=-y, C=1$

$\therefore$ The envelope is $B^2-4AC=0$

$\Rightarrow \quad y^2-4x=0$

$\Rightarrow \quad y^2=4x$.

2. Find the envelope of the family of lines

$y=mx\pm\sqrt{a^2m^2-b^2}$, where *m* is the parameter.

Soln: Given the family is $y=mx\pm\sqrt{a^2m^2-b^2}$, *m* is the parameter.

$\Rightarrow \quad y-mx \ =\pm\sqrt{a^2m^2-b^2}$

$\Rightarrow (y-mx)^2 \ =a^2m^2-b^2$

$\Rightarrow y^2-2mxy+m^2x^2=a^2m^2-b^2$

$\Rightarrow m^2\left(x^2-a^2\right)-2mxy+(y^2+b^2)=0$

This is a quadratic in *m*

Here $A = x^2 - a^2,\ B = -2xy,\ C = y^2 + b^2$

$\therefore$ The envelope is $B^2 - 4AC = 0$

$$\Rightarrow\ 4x^2y^2 - 4(x^2 - a^2)(y^2 + b^2) = 0$$

$$\Rightarrow x^2y^2 - (x^2y^2 + b^2x^2 - a^2y^2 - a^2b^2) = 0 \qquad [\div \text{by } 4]$$

$$\Rightarrow\ b^2x^2 - a^2y^2 - a^2b^2 = 0$$

$$\Rightarrow b^2x^2 - a^2y^2 = a^2b^2$$

$$\Rightarrow\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

3. Find the envelope of the straight lines represented by

$x\cos\alpha + y\sin\alpha = a\ \sec\alpha$, where α is a parameter.

Soln: Given $x\cos\alpha + y\sin\alpha = a\ \sec\alpha$

Dividing by $\cos\alpha$, $\quad x + y\tan\alpha = a\ \sec^2\alpha$

$$= a(1 + \tan^2\alpha)$$

$\Rightarrow\ \ a\ \tan^2\alpha - y\tan\alpha + (a - x) = 0$,

which is a quadratic in $\tan\alpha$

Here $A = a,\quad B = -y,\quad C = a - x$

$\therefore$ The envelope is $B^2 - 4AC = 0 \quad \Rightarrow \quad y^2 - 4a(a - x) = 0,$ which is the envelope.

is the envelope and it is an ellipse.

4. Find the envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$, where the parameters a and b are related by the equation $a^n+b^n=c^n, c$ being a constant.

Soln: Given $\frac{x}{a}+\frac{y}{b}=1$ … (1)

$$a^n+b^n=c^n \quad \ldots (2)$$

Differentiating (1) and (2) with respect to a, treating b as a function of a,

we get

$(1)\Rightarrow \frac{x}{a}+\frac{y}{b}=1$ Diff. w.r.to a

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0 \Rightarrow \frac{db}{da}=-\frac{b^2x}{a^2y} \quad \ldots (3)$$

and $(2)\Rightarrow a^n+b^n=c^n$ Diff. w.r.to a

$$n\,a^{n-1}+n\,b^{n-1}\cdot\frac{db}{da}=0 \Rightarrow \frac{db}{da}=-\frac{a^{n-1}}{b^{n-1}} \quad \ldots (4)$$

From (3) and (4); $\frac{-b^2x}{a^2y}=-\frac{a^{n-1}}{b^{n-1}}$

$$\Rightarrow \quad \frac{x}{a^{n+1}}=\frac{y}{b^{n+1}}$$

$$\Rightarrow \quad \frac{\frac{x}{a}}{a^n}=\frac{\frac{y}{b}}{b^n}=\frac{\frac{x}{a}+\frac{y}{b}}{a^n+b^n}=\frac{1}{c^n} \quad \text{[Using (2)]}$$

$$\therefore \quad \frac{x}{a^{n+1}}=\frac{1}{c^n}$$

$\Rightarrow \quad a = \left(c^{n}\, x\right)^{\frac{1}{n+1}}$

SImilarly $\quad b = \left(c^{n}\, y\right)^{\frac{1}{n+1}}$

Substitute in (1)

$$\frac{x}{c^{n/n+1} \cdot x^{1/n+1}} + \frac{y}{c^{n/n+1} \cdot y^{1/n+1}} = 1$$

$$\Rightarrow \quad x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}},$$

which is the required envelope.