



UNIT 3- DIFFERENTIAL CALCULUS

Evolute

1. Find the equation of the rectangular hyperbola $xy = c^2$.

Soln: Given $xy = c^2$... (1)

Let $P\left(ct, \frac{c}{t}\right)$ be any point on (1)

$$y = \frac{c^2}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2} \Rightarrow y_1 = -\frac{c^2}{x^2}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{2c^2}{x^3} \Rightarrow y_2 = \frac{2c^2}{x^3}$$

$$\therefore \text{At } \left(ct, \frac{c}{t}\right), y_1 = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$$

$$\text{and } y_2 = \frac{2c^2}{c^3t^3} = \frac{2}{ct^3}$$

$$\therefore y_1 = -\frac{1}{t^2}, y_2 = \frac{2}{ct^3}$$

The centre of curvature (\bar{x}, \bar{y}) at P is given by

$$\bar{x} = x - y_1 \left(\frac{1 + y_1^2}{y_2} \right)$$

$$= ct - \frac{-\frac{1}{t^2} \left(1 + \frac{1}{t^4} \right)}{\frac{2}{ct^3}} = ct + \frac{ct}{2} \left(1 + \frac{1}{t^4} \right)$$



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$$= \frac{3ct}{2} + \frac{c}{2t^3}$$

$$\bar{x} = \frac{c}{2t^3}(3t^4 + 1) \quad \dots (2)$$

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2} \right)$$

$$= \frac{c}{t} + \frac{1 + \frac{1}{t^4}}{\frac{ct^3}{2}} = \frac{c}{t} + \frac{c}{2t}(t^4 + 1) = \frac{3c}{2t} + \frac{ct^4}{2t}$$

$$= \frac{c}{2t^3}(3t^2 + t^6) \dots (3)$$

$$\therefore \bar{x} + \bar{y} = \frac{c}{2t^3}[3t^4 + 1 + 3t^2 + t^6]$$

$$= \frac{c}{2t^3}[1 + 3t^2 + 3t^4 + t^6] = \frac{c}{2t^3}(1 + t^2)^3$$

$$\Rightarrow \bar{x} + \bar{y} = \frac{c}{2} \left(\frac{t^2 + 1}{t} \right)^3$$

$$\Rightarrow (\bar{x} + \bar{y})^{1/3} = \left(\frac{c}{2} \right)^{1/3} \left(\frac{t^2 + 1}{t} \right) \dots (4)$$

$$\text{Also } \bar{x} - \bar{y} = \frac{c}{2t^3}[3t^4 + 1 - 3t^2 - t^6]$$

$$= \frac{c}{2t^3}[1 - 3t^2 + 3t^4 - t^6] = \frac{c}{2t^3}(1 - t^2)^3$$



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$$= \frac{c}{2} \left(\frac{1-t^2}{t} \right)^3$$

$$\Rightarrow (\bar{x} - \bar{y})^{1/3} = \left(\frac{c}{2} \right)^{1/3} \left(\frac{1-t^2}{t} \right) \dots (5)$$

Eliminating t from (4) and (5), we get the equation of the evolute.

$$(\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = \left(\frac{c}{2} \right)^{2/3} \left[\left(\frac{1+t^2}{t} \right)^2 - \left(\frac{1-t^2}{t} \right)^2 \right]$$

$$= \left(\frac{c}{2} \right)^{2/3} \left[\frac{(1+t^2)^2 - (1-t^2)^2}{t^2} \right]$$

$$= c^{2/3} \cdot 2^{2-\frac{2}{3}} = c^{2/3} 2^{4/3} = c^{2/3} (2^2)^{2/3} = c^{2/3} (4)^{2/3}$$

$$\Rightarrow (\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = (4c)^{2/3}$$

\therefore Locus of (\bar{x}, \bar{y}) is $(x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3}$, which is the equation of the evolute.

2. Show that the evolute of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ is another cycloid.

Soln: Let ' θ ' be any point on the cycloid.

Given, $x = a(\theta - \sin \theta)$

$$\therefore \frac{dx}{d\theta} = a(1 - \cos \theta) = 2a \sin^2 \frac{\theta}{2}$$

$$\therefore \frac{dy}{d\theta} = a \sin \theta = 2a \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$



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$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2a \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2a \sin^2 \frac{\theta}{2}}$$

$$= \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{d\theta} \left(\cot \frac{\theta}{2} \right) \frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{2} \cdot \frac{1}{2a \sin^2 \frac{\theta}{2}}$$

$$= \frac{-\operatorname{cosec}^4 \frac{\theta}{2}}{4a}$$

$$\therefore y_1 = \cot \frac{\theta}{2}, \quad y_2 = -\frac{\operatorname{cosec}^4 \frac{\theta}{2}}{4a}$$

The centre of curvature (\bar{x}, \bar{y}) at θ is given by

$$\bar{x} = x - y_1 \left(\frac{1 + y_1^2}{y_2} \right)$$

$$= a(\theta - \sin \theta) - \cot \frac{\theta}{2} \left(\frac{1 + \cot^2 \frac{\theta}{2}}{\frac{-\operatorname{cosec}^4 \frac{\theta}{2}}{4a}} \right)$$



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$$= a(\theta - \sin \theta) + \frac{4a \cos \frac{\theta}{2} \operatorname{cosec}^2 \frac{\theta}{2}}{\sin \frac{\theta}{2} \operatorname{cosec}^4 \frac{\theta}{2}}$$

$$= a(\theta - \sin \theta) + 4a \cos \frac{\theta}{2} \cdot \sin \frac{\theta}{2}$$

$$= a(\theta - \sin \theta) + 2a \cdot \sin \theta$$

$$= a\theta + a \sin \theta$$

$$\Rightarrow \bar{x} = a(\theta + \sin \theta) \quad \dots (1)$$

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2} \right)$$

$$= a(1 - \cos \theta) + \frac{1 + \cot^2 \frac{\theta}{2}}{\left(\frac{-\operatorname{cosec}^4 \frac{\theta}{2}}{4a} \right)}$$

$$= a(1 - \cos \theta) - 4a \frac{\operatorname{cosec}^2 \frac{\theta}{2}}{\operatorname{cosec}^4 \frac{\theta}{2}}$$

$$= a(1 - \cos \theta) - 4a \sin^2 \frac{\theta}{2}$$

$$= a \left(2 \sin^2 \frac{\theta}{2} \right) - 4a \sin^2 \frac{\theta}{2}$$

$$= -2a \sin^2 \frac{\theta}{2}$$



SNS COLLEGE OF TECHNOLOGY
(An Autonomous Institution)
Coimbatore-641035.



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$$\Rightarrow \bar{y} = -a(1 - \cos \theta) \dots (2)$$

\therefore The locus of (\bar{x}, \bar{y}) is given by the parametric equations

$x = a(\theta + \sin \theta)$, $y = -a(1 - \cos \theta)$, which is the another cycloid.