



UNIT 3- DIFFERENTIAL CALCULUS

Curvature in Cartesian coordinates

Curvature: Curvature of a curve is a measure of rate of change of bendiness.

Radius of curvature to a curve at a point is denoted by ρ and is the reciprocal of the curvature at that

point. Thus $\rho = \frac{1}{k}$

Cartesian form: If $y = f(x)$, then

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

If $x = f(y)$, then

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)}$$

1. Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$.

SOLN: The radius of curvature at the given point is

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)}$$

Equation of the given curve is $x^3 + y^3 = 3axy$,

Differentiating the equation of the curve w.r.to x , we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a \left[x \frac{dy}{dx} + y \right]$$

$$i. e., 3(y^2 - ax) \frac{dy}{dx} = 3(ay - x^2)$$



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore-641035.



UNIT 3- DIFFERENTIAL CALCULUS

Curvature in Cartesian coordinates

$$\therefore \frac{dy}{dx} = \frac{(ay - x^2)}{(y^2 - ax)} \quad \dots (1)$$

Again differentiating w.r.to. 'x', we get

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax) \left(a \frac{dy}{dx} - 2x \right) - (ay - x^2) \left(2y \frac{dy}{dx} - a \right)}{(y^2 - ax)^2} \quad \dots (2)$$

$$\therefore \left(\frac{dy}{dx} \right)_{\left(\frac{3a}{2}, \frac{3a}{2} \right)} = \frac{\left(\frac{3a^2}{2} - \frac{9a^2}{4} \right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2} \right)} = -1$$

$$\left(\frac{d^2y}{dx^2} \right)_{\left(\frac{3a}{2}, \frac{3a}{2} \right)} = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2} \right) (-a - 3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4} \right) (-3a - a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2} \right)^2} = -1$$

$$= \frac{\frac{3a^2}{4} (-4a) - \left(\frac{-3a^2}{4} \right) (-4a)}{\left(\frac{3a^2}{4} \right)^2}$$

$$= \frac{-3a^3 - 3a^3}{\left(\frac{9a^4}{16} \right)}$$

$$= \frac{-6a^3 \times 16}{9a^4} = \frac{-32}{3a}$$

Hence, the radius of curvature at point $\left(\frac{3a}{2}, \frac{3a}{2} \right)$ is

$$\rho = \frac{[1 + (-1)^2]^{\frac{3}{2}}}{\left(\frac{-32}{3a} \right)} = \frac{-\left(2^{\frac{3}{2}} \times 3a \right)}{32}$$

$$\rho = \frac{-3\sqrt{2}a}{16}$$

i. e., $\rho = \frac{3\sqrt{2}a}{16}$, since ρ is always non - negative.



UNIT 3- DIFFERENTIAL CALCULUS

Curvature in Cartesian coordinates

2. Find the radius of curvature at the point $\left(\frac{1}{4}, \frac{1}{4}\right)$ on the curve $\sqrt{x} + \sqrt{y} = 1$

SOLN: Equation of the given curve is $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$.

Differentiating w.r.to x , we get

$$\begin{aligned}\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} &= 0 \\ \text{i.e., } \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{\sqrt{y}}{\sqrt{x}}\end{aligned}$$

Again differentiating w.r.to x , we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-\left[\sqrt{x}\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} - \sqrt{y}\frac{1}{2}x^{-\frac{1}{2}}\right]}{x} \\ &= -\frac{\left[\frac{\sqrt{x}}{2\sqrt{y}}\frac{dy}{dx} - \frac{\sqrt{y}}{2\sqrt{x}}\right]}{x} \\ &= -\frac{\left[\frac{\sqrt{x}}{2\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) - \frac{\sqrt{y}}{2\sqrt{x}}\right]}{x} \\ &= \frac{-\left(\frac{1}{2} + \frac{\sqrt{y}}{2\sqrt{x}}\right)}{x} \\ &= \frac{-(\sqrt{x} + \sqrt{y})}{2x\sqrt{x}}\end{aligned}$$

$$\text{Now, } \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4}, \frac{1}{4}\right)} = \frac{-\left(\frac{1}{\sqrt{4}}\right)}{\left(\frac{1}{\sqrt{4}}\right)} = -1$$



UNIT 3- DIFFERENTIAL CALCULUS

Curvature in Cartesian coordinates

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{1}{4}, \frac{1}{4}\right)} = \frac{\left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{4}}\right)}{2 \cdot \frac{1}{4} \cdot \frac{1}{\sqrt{4}}} = \frac{\left(\frac{2}{\sqrt{4}}\right)}{\left(\frac{1}{2\sqrt{4}}\right)} = 4$$

Hence, the radius of curvature at the point $\left(\frac{1}{4}, \frac{1}{4}\right)$ is

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)} \text{ at } \left(\frac{1}{4}, \frac{1}{4}\right) = \frac{[1 + (-1)^2]^{\frac{3}{2}}}{4} = \frac{2^{\frac{3}{2}}}{4} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

3. Find the radius of curvature at $x = c$ on the curve $xy = c^2$.

Soln:

Given equation of a curve is $xy = c^2$... (1)

Since $x = c, cy = c^2 \Rightarrow y = c$

\therefore we have to find the radius of curvature at (c, c) on $xy = c^2$

Differentiating equation (1) w.r.to.x, we get

$$\begin{aligned} x \frac{dy}{dx} + y &= 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{y}{x} \quad \dots (2) \\ \therefore \left(\frac{dy}{dx}\right)_{c,c} &= \frac{-c}{c} = -1 \end{aligned}$$

Differentiating (2) w.r.to. x, we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{\left[x \frac{dy}{dx} - y \cdot 1\right]}{x^2} = -\frac{\left[x \cdot \left(-\frac{y}{x}\right) - y\right]}{x^2} \\ &= \frac{2y}{x^2} \end{aligned}$$



UNIT 3- DIFFERENTIAL CALCULUS

Curvature in Cartesian coordinates

$$\therefore \left(\frac{d^2y}{dx^2} \right)_{(c,c)} = \frac{2c}{c^2} = \frac{2}{c}$$

\therefore The radius of curvature at (c,c) is

$$\begin{aligned} \rho &= \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2} \right)} \text{ at } (c, c) \\ &= \frac{[1 + (-1)^2]^{\frac{3}{2}}}{\left(\frac{2}{c} \right)} = \frac{2^{\frac{3}{2}}}{\left(\frac{2}{c} \right)} = 2\sqrt{2} \cdot \frac{c}{2} \end{aligned}$$

i.e. $\rho = c\sqrt{2}$