



A Partial differential Equation in which the partial derivatives occur are of the first degree, is said to be linear; otherwise it is said to be non-linear.

Standard Types:

TYPE 1: $F(p, q) = 0$

TYPE 2: $z = px + qy + f(p, q)$ [Clairaut's form]

TYPE 3: $f(z, p, q) = 0$

TYPE 4: $f_1(x, p) = f_2(y, q)$

TYPE I working Rule

1). Let $z = ax + by + c$ be the complete integral

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$$

2). Put $b = \phi(a)$ for general solution.

3). There is no singular integral.

1. Solve $p + q = pq \rightarrow (1)$

Soln.:

Let $z = ax + by + c \rightarrow (2)$

Complete Integral:

Diff. partially w.r. to 'x' and 'y'

$$\frac{\partial z}{\partial x} = a \quad \left| \quad \frac{\partial z}{\partial y} = b \right.$$

$$p = a \quad \left| \quad q = b \right.$$

Subs. the above values in (1), we get

$$a+b=ab$$

$$a=ab-b$$

$$a=b(a-1) \Rightarrow b = \frac{a}{a-1}$$

The complete integral is,

$$z = ax + \left(\frac{a}{a-1}\right)y + c \rightarrow (3)$$

Singular Integral:

~~diff.~~ (3) partially w.r. to a and c and equal to zero.

$$\frac{\partial z}{\partial a} = x + \left[\frac{(a-1)(1) - a(1)}{(a-1)^2} \right] y = 0$$

$$\frac{\partial z}{\partial c} = 1 \neq 0$$

There is no singular integral.

General Integral:

put $c = \phi(a)$ in (3)

$$z = ax + \left(\frac{a}{a-1}\right)y + \phi(a) \rightarrow (4)$$

~~diff.~~ (4) partially w.r. to a

$$\frac{\partial z}{\partial a} = x + \left[\frac{(a-1)(1) - a(1)}{(a-1)^2} \right] y + \phi'(a) = 0 \rightarrow (5)$$

Eliminate a b/w (4) and (5), we get the general solution.

Q]. Solve $\sqrt{p} + \sqrt{q} = 1$

Soln: $\sqrt{p} + \sqrt{q} = 1 \rightarrow (1)$

Let $z = ax + by + C$

Complete Integral:

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b$$

Subs. the above values in (1), we get

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1 - \sqrt{a})^2$$

The complete integral is,

$$z = ax + (1 - \sqrt{a})^2 y + c \rightarrow (2)$$

Singular Integral:

$$\frac{\partial z}{\partial a} = x + 2(1 - \sqrt{a}) \left(\frac{-1}{2\sqrt{a}} \right) y = 0$$

$$\frac{\partial z}{\partial c} = 1 \neq 0$$

There is no singular integral.

General Integral:

$$\text{Put } c = \phi(a) \text{ in (2)}$$

$$z = ax + (1 - \sqrt{a})^2 y + \phi(a) \rightarrow (3)$$

Diff. (3) partially w.r. to 'a'

$$\frac{\partial z}{\partial a} = x + 2(1 - \sqrt{a}) \left(\frac{-1}{2\sqrt{a}} \right) y + \phi'(a) = 0 \rightarrow (4)$$

Eliminate 'a' b/w (3) and (4),

we get the general solⁿ integral.

TYPE-II Clairaut's Form $z = px + qy + f(p, q)$

Working Rule:

Complete Integral:

Replace $p \rightarrow a$ and $q \rightarrow b$

Singular Integral:

$$\frac{\partial z}{\partial a} = 0 \quad \text{and} \quad \frac{\partial z}{\partial b} = 0$$

General Integral:

Put $b = \phi(a)$ in Complete Integral.

II. Solve $z = px + qy + pq$

Soln.:

Given. $z = px + qy + pq \rightarrow (1)$

Complete Integral:

$$z = ax + by + ab \quad \left[\begin{array}{l} \text{Replace } p \rightarrow a \\ q \rightarrow b \end{array} \right]$$

Singular Integral:

$$\frac{\partial z}{\partial a} = 0 \quad \text{and} \quad \frac{\partial z}{\partial b} = 0$$

$$x + b = 0$$

$$y + a = 0$$

$$b = -x$$

$$a = -y$$

Subs. a & b in (2)

$$z = -yx - xy - y(-x)$$

$$z = -xy$$

General Integral:

Subs. $b = \phi(a)$ in (2),

$$z = ax + \phi(a)y + a\phi(a) \rightarrow (3)$$

wkt $\frac{\partial z}{\partial a} = 0$

$$\Rightarrow x + \phi'(a)y + a\phi'(a) + \phi(a) = 0 \rightarrow (4)$$

Eliminate 'a' b/w (4) & (3), we get

the general soln.

Q. Solve $z = px + qy + p^2 - q^2$

Soln:

Given. $z = px + qy + p^2 - q^2 \rightarrow (1)$

Complete Integral:

$z = ax + by + a^2 - b^2$ Replace $p \rightarrow a$
 $\hookrightarrow (2)$ $q \rightarrow b$

Singular Integral:

$$\left. \begin{aligned} \frac{\partial z}{\partial a} &= 0 \\ x + 2a &= 0 \\ 2a &= -x \\ a &= \frac{-x}{2} \end{aligned} \right\} \begin{aligned} \frac{\partial z}{\partial b} &= 0 \\ y - 2b &= 0 \\ y &= 2b \\ \Rightarrow b &= \frac{y}{2} \end{aligned}$$

Subs. a & b in (2),

$$\begin{aligned} z &= \frac{-x}{2}x + \frac{y}{2}y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2 \\ &= -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4} \end{aligned}$$

$$\begin{aligned} 4z &= -2x^2 + 2y^2 + x^2 - y^2 \\ &= -x^2 + y^2 \end{aligned}$$

$$y^2 - x^2 = 4z$$

General Integral:

Subs. $b = \phi(a)$ in (2)

$$z = ax + \phi(a)y + [\phi(a)]^2 + a^2 \rightarrow (3)$$

wkt $\frac{\partial z}{\partial a} = 0$

$$\Rightarrow x + \phi'(a)y - 2\phi(a)\phi'(a) + 2a = 0 \rightarrow (4)$$

Eliminate 'a' b/w (3) & (4), we get the general soln.

Type - III $f(x, p, q) = 0$

7. solve $p(1+q) = qx \rightarrow (1)$

Soln.:

Let $u = x + ay$

Then $p = \frac{dx}{du}$ and $q = a \frac{dz}{du}$

$$(1) \Rightarrow \frac{dx}{du} \left(1 + a \frac{dx}{du}\right) = a \frac{dx}{du} x$$

$$1 + a \frac{dx}{du} = ax$$

$$a \frac{dx}{du} = ax - 1$$

$$\frac{dx}{du} = \frac{ax-1}{a}$$

$$\frac{du}{dx} = \frac{a}{ax-1}$$

$$du = \frac{a}{ax-1} dx$$

Integrating,

$$u = \int \frac{a}{ax-1} dx$$

$$u = \log(ax-1) + \log c$$

$$x+ay = \log [c(ax-1)]$$

8. Solve $x^2 = 1 + p^2 + q^2$

Soln.:

$$x^2 = 1 + p^2 + q^2 \rightarrow (1)$$

Let $u = x + ay$

$$p = \frac{dx}{du}, \quad q = a \frac{dz}{du}$$

$$(1) \Rightarrow z^2 = 1 + \left(\frac{dz}{du}\right)^2 + \left(u \frac{dz}{du}\right)^2$$

$$z^2 = \left(\frac{dz}{du}\right)^2 (1 + a^2) + 1$$

$$z^2 - 1 = \left(\frac{dz}{du}\right)^2 (1 + a^2)$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2 - 1}{1 + a^2}$$

$$\frac{dz}{du} = \sqrt{\frac{z^2 - 1}{1 + a^2}} = \frac{\sqrt{z^2 - 1}}{\sqrt{1 + a^2}}$$

$$\frac{dz}{\sqrt{z^2 - 1}} = \frac{du}{\sqrt{1 + a^2}}$$

Integrating on both sides,

$$\cosh^{-1} z = \frac{1}{\sqrt{1 + a^2}} u + C$$

$$= \frac{1}{\sqrt{1 + a^2}} (x + ay) + C$$

TYPE - IV $f_1(x, p) = f_2(y, q)$

For this type, there is no singular integral.

7. Solve $q^2 - p = y - x$

Soln.:

Given. $q^2 - y = p - x = k$ (a constant)

$$\begin{array}{l|l} \text{Now } q^2 - y = k & p - x = k \\ q^2 = k + y & p = k + x \\ q = \sqrt{k + y} & \end{array}$$

we know that $x = \int p dx + \int q dy$

$$x = \int (k + x) dx + \int \sqrt{k + y} dy$$

$$= kx + \frac{x^2}{2} + \frac{(k + y)^{3/2}}{3/2} + c$$

$$= kx + \frac{x^2}{2} + \frac{2}{3} (k + y)^{3/2} + c, \text{ which is the complete integral.}$$

8. Solve $\sqrt{p} + \sqrt{q} = x + y$

Soln.:

Given. $\sqrt{p} - x = y - \sqrt{q} = k$

$$\begin{array}{l|l} \text{Now } \sqrt{p} - x = k & y - \sqrt{q} = k \\ \sqrt{p} = k + x & \sqrt{q} = y - k \\ p = (k + x)^2 & q = (y - k)^2 \end{array}$$

we know that

$$x = \int p dx + \int q dy$$

$$z = \int (k+x)^2 dx + \int (y-k)^2 dy$$

$$= \frac{(k+x)^3}{3} + \frac{(y-k)^3}{3} + c, \text{ which is the complete integral}$$

3). Find the complete integral of

$$xp - yq = y^2 - x^2$$

Soln.

Given. $xp + x^2 = y^2 + yq = k$ (a constant)

$$\begin{array}{l|l} \text{Now } xp + x^2 = k & y^2 + yq = k \\ xp = k - x^2 & yq = k - y^2 \\ p = \frac{k - x^2}{x} & q = \frac{k - y^2}{y} \\ p = \frac{k}{x} - x & q = \frac{k}{y} - y \end{array}$$

we know that $z = \int p dx + \int q dy$

$$z = \int \left(\frac{k}{x} - x \right) dx + \int \left(\frac{k}{y} - y \right) dy$$

$$= k \log x - \frac{x^2}{2} + k \log y - \frac{y^2}{2} + c$$

$$= k \log xy - \left(\frac{x^2 + y^2}{2} \right) + c \text{ which is the CI.}$$

Q. Form the PDE from $z = ax + by + \sqrt{a^2 + b^2}$

Soln.:

$$\text{Given } z = ax + by + \sqrt{a^2 + b^2} \rightarrow (1)$$

Differentiate partially w.r. to 'x'

$$\frac{\partial z}{\partial x} = a + 0 + 0$$

$$\Rightarrow p = a \rightarrow (2)$$

Differentiate partially w.r. to 'y'

$$\frac{\partial z}{\partial y} = 0 + b + 0$$

$$\Rightarrow q = b \rightarrow (3)$$

Subst. (2) and (3) in (1),

$$z = px + qy + \sqrt{p^2 + q^2}$$

Q. Form the PDE from $ax^2 + by^2 + z^2 = 1$

Soln.:

$$\text{Given } ax^2 + by^2 + z^2 = 1 \rightarrow (1)$$

Differentiate partially w.r. to 'x'

$$2ax + 0 + 2z \frac{\partial z}{\partial x} = 0$$

$$2ax = -2zP$$

$$a = \frac{-zP}{x}$$

Differentiate partially w.r.t 'y'

$$0 + 2by + 2z \frac{\partial z}{\partial y} = 0$$

$$2by + 2zq = 0$$

$$2by = -2zq$$

$$b = \frac{-zq}{y}$$

Subst. a and b in (1),

$$\frac{-zP}{x} x^2 - \frac{zq}{y} y^2 + z^2 = 1$$

$$-zPx - zqy + z^2 = 1$$

$$z(z - Px - qy) = 1$$

iv).

$$z = f(x+t) + g(x-t)$$

$$p = f'(x+t) + g'(x-t)$$

$$q = f'(x+t) - g'(x-t)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+t) + g''(x-t) \rightarrow (1)$$

$$s = \frac{\partial^2 z}{\partial x \partial t} = f''(x+t) - g''(x-t) \rightarrow (2)$$

$$t = \frac{\partial^2 z}{\partial t^2} = f''(x+t) + g''(x-t) \rightarrow (3)$$

From
(1) and (3),

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$$