Properties of Fourier Transforms

Property I: Linear Property

$$F[a f(x) \pm b g(x)] = a F(s) \pm b G(s)$$

Proof: $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$

$$F[a f(x) \pm b g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a f(x) \pm b g(x)] e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a f(x) e^{isx} dx \pm \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b g(x) e^{isx} dx.$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \pm \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cdot e^{isx} dx.$$

$$= a F(s) \pm b G(s)$$

Property 2: Change of Scale Property

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right), \quad a > 0.$$

Proof:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx.$$

put $ax = y$ | $x = -\infty$; $y = -\infty$

$a\,dx = dy$ | $x = \infty$; $y = \infty$

$dx = \frac{dy}{a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{is\left(\frac{y}{a}\right)} \frac{dy}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{i\left(\frac{s}{a}\right)y} dy = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right), \quad a > 0$$

Property 8 : Shifting Theorem (or) Property

(i) $F[f(x-a)] = e^{ias} F(s)$

(ii) $F[e^{iax} f(x)] = F(s+a)$

Proof: (i) $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

put $x-a = y$, $dx = dy$ | $x = -\infty$; $y = -\infty$, $x = \infty$; $y = \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) . e^{is(y+a)} . dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) . e^{isy} . e^{isa} dy.$$

$$= \frac{e^{isa}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{isy} dy.$$

$$F[f(x-a)] = \frac{e^{isa}}{\sqrt{2\pi}} F(s)$$

(ii) $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx$$

$$F[e^{iax} f(x)] = F(s+a).$$

Note: $F[e^{-iax} f(x)] = F(s-a)$

Property 4: Modulation Theorem or Property:

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a)+F(s-a)]$$

Property 5: $F[x^n f(x)] = (-i)^n \dfrac{d^n F(s)}{ds^n}$

Property 6: $F[f'(x)] = (-is) F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

Property 7: $F\left[\int_a^x f(x)\,dx\right] = \dfrac{F(s)}{(-is)}$

Property 8: $F[\overline{f(x)}] = \overline{F(-s)}$

Property 9: $F[f(-x)] = F(-s)$

Property 10: $F[\overline{f(-x)}] = \overline{F(s)}$

Convolution of Two Functions:

The convolution of $f(x)$ and $g(x)$ is defined as

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\,dt$$

$$f*g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\,dt$$

Convolution Theorem for Fourier Transforms:

If $F(s)$ and $G(s)$ are the fourier transforms of $f(x)$ and $g(x)$ respectively then the fourier Transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier Transform.

$$F[(f*g)x] = F(s).G(s).$$

Properties of Fourier Sine & Cosine Transforms:

Property 1: Linear Property

a) $F_s[af(x)+bg(x)] = aF_s(s)+bG_s(s)$

b) $F_c[af(x)+bg(x)] = aF_c(s)+bG_c(s)$

Property 2: Modulation Property

(i) $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a)+F_s(s-a)]$

(ii) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a)+F_c(s-a)]$

(iii) $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a)-F_c(s+a)]$

(iv) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(s+a)+F_s(a-s)]$

Property 3: $F_s[f'(x)] = -sF_c(s)$

Property 4: $F_c[f'(x)] = -\sqrt{\frac{2}{\pi}}f(0)+sF_s(s)$

Property 5: $F_s[x.f(x)] = -\frac{d}{ds}F_c[f(x)]$ ⓧ

Property 6: $F_c[x.f(x)] = \frac{d}{ds}F_s[f(x)]$ ⓧ

Property 7: $F_s[f(ax)] = \frac{1}{a}F_s\left(\frac{s}{a}\right)$

$F_c[f(ax)] = \frac{1}{a}F_c\left(\frac{s}{a}\right)$ ⓧ

Property 8: (i) $\int_0^\infty F_c[f(x)].G_c[g(x)]\,ds = \int_0^\infty f(x).g(x)\,dx$

(ii) $\int_0^\infty F_s[f(x)].G_s[g(x)]\,ds = \int_0^\infty f(x)g(x)\,dx.$ ⓧ

Parseval's Identity:

1) $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(s)|^2ds$

2) $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F_s(s)|^2ds$

3) $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F_c(s)|^2ds$