Self - Reciprocal

Definition:

If $F[f(x)]$ is $f(s)$, then $f(x)$ is self-reciprocal under Fourier Transform.

Problems:

1). Find the Fourier Transform of $e^{-a^2x^2}$, $a>0$. Hence show that $e^{-x^2/2}$ is self-reciprocal under Fourier Transform.

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$$

$$F[e^{-a^2x^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} \cdot e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2+isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right)}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} \cdot e^{\left(\frac{is}{2a}\right)^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} \cdot e^{-s^2/4a^2} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx.$$

Put $t = ax - \frac{is}{2a}$

$\frac{dt}{dx} = a - 0$

$dt = dx$

$\frac{dt}{a}$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \frac{1}{a} \int_{-\infty}^{\infty} e^{-t^2} \cdot dt$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \frac{1}{a}(\sqrt{\pi}) = \frac{e^{-s^2/4a^2}}{\sqrt{2}\ a}$$

formula:

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

$$F[e^{-a^2x^2}] = \frac{e^{-s^2/4a^2}}{\sqrt{2}\ a}.$$

put $a = 1/\sqrt{2}$; $a^2 = 1/2$

$$F[e^{-x^2/2}] = e^{-s^2/2}$$

$\therefore e^{-x^2/2}$ is self-reciprocal under Fourier Transforms

2) Show that the Fourier Transform of $f(x) = e^{-x^2/2}$ is $e^{-s^2/2}$

(or) Show that $e^{-x^2/2}$ is self-reciprocal under Fourier Transform.

Solution:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2 + isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} - isx\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx + (is)^2 - (is)^2)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx + (is)^2)} \cdot e^{-(is)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x - is)^2} \cdot e^{\frac{-(is)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x - is)^2} \cdot e^{s^2/2} dx = \frac{e^{s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x - is)^2} dx$$

$$= \frac{e^{s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x - is}{\sqrt{2}}\right)^2} dx$$

Let $t = \frac{x - is}{\sqrt{2}}$

$dt = \frac{dx}{\sqrt{2}}$

$\Rightarrow \sqrt{2}.\, dt = dx$

$$\therefore F(s) = \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\ldots$$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \cdot \sqrt{2} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{e^{-s^2/2}}{\sqrt{\pi}} (\sqrt{\pi})$$

$$\boxed{\therefore F(s) = e^{-s^2/2}}$$

$\therefore f(x) = e^{-x^2/2}$ is a self-reciprocal function.

Problem:

1) Find the Fourier transform of $f(x) = xe^{-x}$, $0 \le x \le \infty$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-x} . e^{isx} dx \qquad e^{-\infty} = 0$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-x(1-is)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[x \frac{e^{-x(1-is)}}{-(1-is)} - \frac{e^{-x(1-is)}}{(1-is)^2} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - 0 - \left(0 - \frac{e^0}{(1-is)^2} \right) \right] = \frac{1}{\sqrt{2\pi}} \left(\frac{+1}{(1-is)^2} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{+1}{(1-is)^2} \right) \qquad \boxed{F(s) = \frac{1}{\sqrt{2\pi}(1-is)^2}}$$

Scanned by CamScanner