Partial fractions method:

(1) Find $Z^{-1}\left[\frac{10z}{(z-1)(z-2)}\right]$

Soln:

$$X(z) = \frac{10z}{(z-1)(z-2)}$$

$$\frac{X(z)}{z} = \frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow (1)$$

$$10 = A(z-2) + B(z-1)$$

Put $z=1 \Rightarrow -A = 10 \Rightarrow A = -10$

Put $z=2 \Rightarrow B = 10$

$$\frac{X(z)}{z} = \frac{-10}{z-1} + \frac{10}{z-2}$$

$$X(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

$$Z\{x(n)\} = 10\left[\frac{z}{z-2}\right] - 10\left[\frac{z}{z-1}\right]$$

$$x(n) = 10Z^{-1}\left[\frac{z}{z-2}\right] - 10Z^{-1}\left[\frac{z}{z-1}\right]$$

$$= 10(2^n) - 10$$

$$\boxed{x(n) = 10(2^n - 1)}$$

(2) Find $Z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right]$

Soln: Let $X(z) = \frac{z^2}{(z+2)(z^2+4)}$

$$\frac{X(z)}{z} = \frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4} \rightarrow (1)$$

$$z = A(z^2+4) + (Bz+C)(z+2)$$

Put $z = -2 \Rightarrow -2 = 8A \Rightarrow A = -1/4$

Put $z = 0 \Rightarrow 0 = 4A + 2C \Rightarrow 2C = -4 \times \frac{-1}{4} = 1$

$\Rightarrow C = 1/2$

Equating z^2 on both sides,

$0 = A + B \Rightarrow B = -A = 1/4$

$$\frac{X(z)}{z} = \frac{\frac{-1}{4}}{z+2} + \frac{\frac{1}{4}z + \frac{1}{2}}{z^2+4}$$

$$= -\frac{1}{4} \cdot \frac{1}{z-(-2)} + \frac{1}{4} \cdot \frac{z}{z^2+4} + \frac{1}{2} \cdot \frac{1}{z^2+4}$$

$$Z[x(n)] = X(z) = -\frac{1}{4}\left[\frac{z}{z-(-2)}\right] + \frac{1}{4}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{2}\frac{z}{z^2+4}$$

$$x(n) = \frac{-1}{4} z^{-1}\left[\frac{z}{z-(-2)}\right] + \frac{1}{4} z^{-1}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{4} z^{-1}\left[\frac{2z}{z^2+4}\right]$$

$$= \frac{-1}{4}(-2)^n + \frac{1}{4} 2^n \cos\frac{n\pi}{2} + \frac{1}{4} 2^n \sin\frac{n\pi}{2}$$

$$\left[Z\left[a^n \cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+a^2} \quad Z\left[a^n \sin\frac{n\pi}{2}\right] = \frac{az}{z^2+a^2} \right]$$

(3) Find $z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right]$ using partial fractions.

Soln:

Let $X(z) = \frac{z^3}{(z-1)^2(z-2)}$

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$z^2 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

Put $z = 1 \Rightarrow 1 = 0 + B(-1) \Rightarrow B = -1$

Put $z = 2 \Rightarrow 4 = 0 + 0 + C \Rightarrow C = 4$

Put $z = 0 \Rightarrow 0 = A(-1)(-2) + B(-2) + C(-1)^2 \Rightarrow A = -3$

$$\therefore \frac{X(z)}{z} = \frac{-3}{z-1} - \frac{1}{(z-1)^2} + \frac{4}{z-2}$$

③ Find $z\left[a^n \sin n\theta\right]$

Soln: We know that $z\left[a^n f(n)\right] = F\left[\frac{z}{a}\right]$

$$z\left[a^n \sin n\theta\right] = \left[z(\sin n\theta)\right]_{z\to z/a}$$

$$= \left[\frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}\right]_{z\to z/a}$$

$$= \frac{\frac{z}{a}\sin\theta}{\frac{z^2}{a^2} - 2\frac{z}{a}\cos\theta + 1} = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

④ Find $z\left[a^n n\right]$

$z\left[a^n f(n)\right] = \left[z[f(n)]\right]_{z\to z/a}$

Soln: We know that $z\left[a^n f(n)\right] = F\left[\frac{z}{a}\right]$

$$z\left[a^n n\right] = \left[z[n]\right]_{z\to z/a}$$

$$= \left[\frac{z}{(z-1)^2}\right]_{z\to z/a} \qquad \left\{\because z(n) = \frac{z}{(z-1)^2}\right\}$$

$$= \frac{z/a}{(z/a - 1)^2} = \frac{z/a}{\left(\frac{z-a}{a}\right)^2} = \frac{az}{(z-a)^2}$$

⑤ Find $z\left[\frac{a^n}{n!}\right]$

Soln: We know that $z\left[a^n f(n)\right] = F\left[\frac{z}{a}\right]$

$$z\left[a^n \cdot \frac{1}{n!}\right] = \left[z\left[\frac{1}{n!}\right]\right]_{z\to z/a}$$

$$= \left[e^{1/z}\right]_{z\to z/a} \qquad \left[\because z\left[\frac{1}{n!}\right] = e^{1/z}\right]$$

$$= e^{\frac{1}{(z/a)}}$$

$$= e^{a/z}$$

⑥ Find $z\left[\frac{a^n}{n}\right]$

Soln: We know that $z\left[a^n f(n)\right] = F\left[\frac{z}{a}\right]$

$$z\left[a^n \cdot \frac{1}{n}\right] = \left[z\left[\frac{1}{n}\right]\right]_{z\to z/a}$$

$$X(z) = \frac{-3z}{z-1} - \frac{z}{(z-1)^2} + 4\frac{z}{z-2}$$

$$z\{x(n)\} = X(z) = -3\cdot\frac{z}{z-1} - \frac{z}{(z-1)^2} + 4\frac{z}{z-2}$$

$$x(n) = -3z^{-1}\left[\frac{z}{z-1}\right] - z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -3(1)^n - n + 4(2)^n.$$

Inverse of z-transform by inverse integral method (cauchy's residue theorem):

$$\int_C X(z)\, z^{n-1}\, dz = 2\pi i \cancel{\oint_c X(z) z^{n-1} dz} \left[\text{Sum of the residues of } X(z)\, z^{n-1} \text{ at the isolated Singularities}\right]$$

i.e., $x(n)$ = Sum of the residues of $X(z)\,z^{n-1}$ at the isolated singularities.

① Find $z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$

Soln:

Let $X(z) = \dfrac{10z}{(z-1)(z-2)}$

$$X(z)\,z^{n-1} = \frac{10z}{(z-1)(z-2)}\, z^{n-1} = \frac{10z^n}{(z-1)(z-2)}$$

$z=1$ is a simple pole and $z=2$ is a simple pole.

$$\operatorname*{Res}_{z=1} X(z)\,z^{n-1} = \lim_{z\to 1} (z-1)\frac{10z^n}{(z-1)(z-2)} = \lim_{z\to 1} \frac{10z^n}{z-2}$$

$$= \frac{10(1)^n}{1-2} = -10$$

$$\operatorname*{Res}_{z=2} X(z)\,z^{n-1} = \lim_{z\to 2} (z-2)\frac{10z^n}{(z-1)(z-2)} = \lim_{z\to 2}\frac{10z^n}{z-1}$$

$$= \frac{10\cdot 2^n}{2-1} = 10(2^n)$$

$\therefore$ $x(n)$ = Sum of the residues $= 10(2^n) - 10 = 10(2^n - 1)$

② Find $z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right]$

Soln:

Let $X(z) = \frac{z(z+1)}{(z-1)^3}$

$$X(z)(z^{n-1}) = \frac{z(z+1)}{(z-1)^3} z^{n-1}$$

$$= \frac{z^n(z+1)}{(z-1)^3}$$

$z = 1$ is a pole of order 3.

$$\operatorname*{Res}_{z=1} X(z) z^{n-1} = \operatorname*{Lt}_{z\to 1} \frac{1}{2!} \frac{d^2}{dz^2}\left[(z-1)^3 \frac{z^n(z+1)}{(z-1)^3}\right]$$

$$= \operatorname*{Lt}_{z\to 1} \frac{1}{2} \frac{d^2}{dz^2}\left[z^n(z+1)\right]$$

$$= \frac{1}{2} \operatorname*{Lt}_{z\to 1} \frac{d}{dz}\left[z^n(1) + (z+1)nz^{n-1}\right]$$

$$= \frac{1}{2} \operatorname*{Lt}_{z\to 1} \frac{d}{dz}\left[z^n + n(z+1)z^{n-1}\right]$$

$$= \frac{1}{2} \operatorname*{Lt}_{z\to 1}\left[nz^{n-1} + n(z+1)(n-1)z^{n-2} + nz^{n-1}(1)\right]$$

$$= \frac{1}{2}\left[n(1)^{n-1} + n\,2\,(n-1)(1)^{n-2} + n(1)^{n-1}\right]$$

$$= \frac{1}{2}\left[2n + 2n(n-1)\right]$$

$$= n + n(n-1)$$

$$= n^2$$

$\therefore x(n) =$ sum of the residues $= n^2$.

② Evaluate $z^{-1}\left[(z-5)^{-3}\right]$ for $|z|>5$. (16)

Soln: $X(z) = \frac{1}{(z-5)^3}$

$$X(z).z^{n-1} = \frac{z^{n-1}}{(z-5)^3}$$

$z=5$ is a pole of order 3.

$$\therefore \operatorname*{Res}_{z=5} X(z)z^{n-1} = \operatorname*{Lt}_{z\to 5} \frac{1}{2!}\frac{d^2}{dz^2} \frac{(z-5)^3 z^{n-1}}{(z-5)^3}$$

$$= \operatorname*{Lt}_{z\to 5} \frac{1}{2}(n-1)(n-2) z^{n-3}$$

$$= \frac{1}{2}(n-1)(n-2)5^{n-3}$$

③ Find $z^{-1}\left[\frac{3z^2+z}{z^3-3z^2+4}\right]$

Let $X(z) = \frac{3z^2+z}{z^3-3z^2+4}$