$$Z\left[\frac{1}{(n+1)(n+2)}\right] = z\log\frac{z}{z-1} - z^2\left[-\log\left(1-\frac{1}{z}\right)-\frac{1}{z}\right]$$

$$= z\log\frac{z}{z-1} + z^2\log\left(\frac{z-1}{z}\right)+z$$

$$= z\log\left(\frac{z}{z-1}\right) - z^2\log\left(\frac{z}{z-1}\right)+z$$

$$= (z-z^2)\log\left(\frac{z}{z-1}\right)+z$$

First Shifting theorem: 1. $Z\{f(n)\} = F(z)$, $Z\{a^n f(n)\} = F(z/a)$

2.

(1) If $Z\{f(t)\} = F(z)$, then $Z\{e^{-at} f(t)\} = F[ze^{aT}]$

(2) If $Z\{f(t)\} = F(z)$, then $Z\{e^{at} f(t)\} = F[ze^{-aT}]$

(3) If $Z\{f(t)\} = F(z)$, then $Z\{a^n f(t)\} = F\left[\frac{z}{a}\right]$

(4) If $Z\{f(n)\} = F(z)$, then $Z\{a^n f(n)\} = F\left[\frac{z}{a}\right]$

$Z\{a^{-n} f(n)\} = F(az)$

Proof:

(1) Given $F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)z^{-n}$

$$Z\{e^{-at} f(t)\} = \sum_{n=0}^{\infty} e^{-anT} f(nT)z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)(e^{aT}z)^{-n}$$

$$= Z[f(t)]_{z\to ze^{aT}}$$

$$= F[ze^{aT}]$$

$$= F[z] \text{ where } z\to ze^{aT}$$

(2) $F(Z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)z^{-n}$

$$Z\{e^{at} f(t)\} = \sum_{n=0}^{\infty} e^{anT} f(nT)z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)(ze^{-aT})^{-n}$$

$$= Z[f(t)]_{z\to (ze^{-aT})}$$

$$= F[ze^{-aT}]$$

$$= F[Z] \text{ where } z\to ze^{-aT}$$

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(iii) $F[z] = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$Z\{a^n f(t)\} = \sum_{n=0}^{\infty} a^n f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \left(\frac{z}{a}\right)^{-n} = F\left(\frac{z}{a}\right)$$

$$= F[Z] \text{ where } Z \to z/a$$

(iv) $F[z] = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z.\{a^n f(n)\} = \sum_{n=0}^{\infty} a^n f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n) \left(\frac{z}{a}\right)^{-n} = F\left(\frac{z}{a}\right)$$

$$= F(z) \text{ where } z \to z/a.$$

Problems:

1. Find $Z[e^{-at} t]$

Soln:

We know that $Z[e^{-at} f(t)] = [Z[f(t)]]_{z \to ze^{aT}}$

$$Z[e^{-at} t] = [Z(t)]_{z \to ze^{aT}}$$

$$= \left[\frac{Tz}{(z-1)^2}\right]_{z \to ze^{aT}}$$

$$= \frac{Tze^{aT}}{(ze^{aT}-1)^2}$$

2. Find $Z[e^{-at} \cos bt]$

Soln:

We know $Z[e^{-at} f(t)] = [Z[f(t)]]_{z \to ze^{aT}}$

$$Z[e^{-at} \cos bt] = [Z[\cos bt]]_{z \to ze^{aT}}$$

$$= \left[\frac{z(z-\cos bT)}{z^2 - 2z\cos bT + 1}\right]_{z \to ze^{aT}}$$

$$= \frac{ze^{aT}[ze^{aT} - \cos bT]}{(ze^{aT})^2 - 2ze^{aT}\cos bT + 1}$$

(3) Find $z[a^n \sin n\theta]$

Soln: We know that $z[a^n f(n)] = F\left[\frac{z}{a}\right]$

$$z[a^n \sin n\theta] = [z(\sin n\theta)]_{z \to z/a}$$

$$= \left[\frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}\right]_{z \to z/a}$$

$$= \frac{\frac{z}{a}\sin\theta}{\frac{z^2}{a^2} - 2\frac{z}{a}\cos\theta + 1} = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

(4) Find $z[a^n n]$

$$z[a^n \cdot f(n)] = [z[f(n)]]_{z \to z/a}$$

Soln: We know that $z[a^n f(n)] = F\left[\frac{z}{a}\right]$

$$z[a^n n] = [z[n]]_{z \to z/a}$$

$$= \left[\frac{z}{(z-1)^2}\right]_{z \to z/a} \quad \left\{\because z(n) = \frac{z}{(z-1)^2}\right\}$$

$$= \frac{z/a}{(z/a - 1)^2} = \frac{z/a}{\left(\frac{z-a}{a}\right)^2} = \frac{az}{(z-a)^2}$$

(5) Find $z\left[\frac{a^n}{n!}\right]$

Soln: We know that $z[a^n f(n)] = F\left[\frac{z}{a}\right]$

$$z\left[a^n \cdot \frac{1}{n!}\right] = \left[z\left[\frac{1}{n!}\right]\right]_{z \to z/a}$$

$$= \left[e^{1/z}\right]_{z \to z/a} \quad \left[\because z\left[\frac{1}{n!}\right] = e^{1/z}\right]$$

$$= e^{\frac{1}{(z/a)}}$$

$$= e^{a/z}$$

(6) Find $z\left[\frac{a^n}{n}\right]$

Soln: We know that $z[a^n f(n)] = F\left[\frac{z}{a}\right]$

$$z\left[a^n \cdot \frac{1}{n}\right] = \left[z\left[\frac{1}{n}\right]\right]_{z \to z/a}$$

(8) Find $z\left[e^{-iat}\right]$ using z-transforms.

Soln:

$$z\left[e^{-iat}\cdot 1\right] = \left[z[1]\right]_{z\to ze^{iaT}}$$

$$= \left[\frac{z}{z-1}\right]_{z\to ze^{iaT}} \qquad \left\{\because z[1] = \frac{z}{z-1}\right\}$$

$$= \left[\frac{ze^{iaT}}{ze^{iaT}-1}\right]$$

Differentiation in the Z-Domain:

(i) $z\left[nf(t)\right] = -z\frac{d}{dz}F(z)$ (ii) $z\left[nf(n)\right] = -z\frac{d}{dz}F(z)$

Proof:

(i) Given: $F[z] = z[f(t)]$

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

$$\frac{d}{dz}\left[F(z)\right] = \sum_{n=0}^{\infty} -nf(nT)z^{-n-1}$$

$$= -\sum_{n=0}^{\infty} nf(nT)\frac{z^{-n}}{z}$$

$$z\frac{d}{dz}F(z) = -\sum_{n=0}^{\infty} nf(nT)z^{-n}$$

$$= -z\left[nf(t)\right]$$

$$\therefore z\left[nf(t)\right] = -z\frac{d}{dz}\left[F(z)\right]$$

(ii) Given: $F(z) = z\left[f(n)\right]$

$$F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$\frac{d}{dz}\left[F(z)\right] = \sum_{n=0}^{\infty}(-n)f(n)z^{-n-1}$$

$$= -\sum_{n=0}^{\infty} nf(n)\frac{z^{-n}}{z}$$

$$z\frac{d}{dz}F(z) = -\sum_{n=0}^{\infty} nf(n)z^{-n} = -z\left[nf(n)\right]$$

$$z\left[nf(n)\right] = -z\frac{d}{dz}F(z)$$

Problems:

$$\boxed{Z[n \cdot f(n)] = -z \cdot \frac{d}{dz} Z[f(n)]}$$

(1) Find the Z-transform of n^2.

Soln:

We know $Z[nf(n)] = -z \dfrac{d}{dz} F(z)$

$$Z[n^2] = Z[n \cdot n] = -z \frac{d}{dz}[Z(n)]$$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right] = -z\left[\frac{-1-z}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}$$

(2) Find the Z-transform of $(n+1)(n+2)$.

Soln:

$$Z[(n+1)(n+2)] = Z[n^2+2n+n+2]$$

$$= Z[n^2+3n+2]$$

$$= Z(n^2) + 3Z(n) + 2Z(1)$$

$$= \frac{z^2+z}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\frac{z}{z-1}$$

$$= \frac{(z^2+z) + 3z(z-1) + 2z(z-1)^2}{(z-1)^3}$$

Unit step Sequence :

The unit step sequence $u(n)$ has values

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Note :

Z-transform of unit step sequence is $\frac{z}{z-1}$ i.e., $Z\{u(n)\} = \frac{z}{z-1}$

Proof :

We know that $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z\{u(n)\} = \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} \quad (\text{by defn of } u(n))$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$= \left[1 - \frac{1}{z}\right]^{-1} = \left(\frac{z-1}{z}\right)^{-1}$$

$$\boxed{Z\{u(n)\} = \frac{z}{z-1}}$$

Problems :

(1) Find $Z[\delta(n-k)]$

Soln:

$$Z[\delta(n-k)] = \sum_{n=0}^{\infty} \delta(n-k) z^{-n} \longrightarrow (1)$$

where $\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$

$$(1) \Rightarrow Z[\delta(n-k)] = \frac{1}{z^k}$$

$$Z[\delta(n-1)] = \frac{1}{z}$$

(2) Find $Z[a^n \delta(n-k)]$

Soln:

$$Z[a^n \delta(n-k)] = Z[\delta(n-k)]_{z \to z/a}$$

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$$

$$\therefore Z[a^n \delta(n-k)] = \left[\frac{1}{z^k}\right]_{z \to z/a} = \frac{1}{\left(\frac{z}{a}\right)^k} = \frac{a^k}{z^k} = \left(\frac{z}{a}\right)^{-k}$$

Initial value theorem:

If $Z[f(t)] = F(z)$, then $f(0) = \underset{z \to \infty}{Lt} F(z)$.

Proof:

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= f(0.T) + \frac{f(1.T)}{z} + \frac{f(2.T)}{z^2} + \cdots$$

$$F(z) = f(0) + \frac{f(T)}{z} + \frac{1}{z^2} f(2T) + \cdots$$

$$\underset{z \to \infty}{Lt} F(z) = \underset{z \to \infty}{Lt} \left[f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \cdots \right]$$

$$= f(0).$$

Final value theorem:

If $Z[f(t)] = F(z)$, then $\underset{t \to \infty}{Lt} f(t) = \underset{z \to 1}{Lt} (z-1) F(z)$

Proof:

$$Z[f(t+T) - f(t)] = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n}.$$

$$Z[f(t+T)] - Z[f(t)] = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n}$$

$$zF(z) - zf(0) - F(z) = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n}$$

Taking limit as $z \to 1$

$$\underset{z \to 1}{Lt} (z-1)F(z) - f(0) = \underset{z \to 1}{Lt} \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n}$$

$$= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)]$$

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$$\underset{z\to 1}{Lt}\ (z-1)F(z)-f(0) = \underset{n\to\infty}{Lt}\ [f(T)+f(0)-f(2T)-f(T) + \dots + f[(n+1)T]-f(nT)]$$

$$= \underset{n\to\infty}{Lt}\ f[(n+1)T]-f(0)$$

$$\underset{z\to 1}{Lt}\ (z-1)F(z)-f(0) = f(\infty)-f(0)$$

$$\therefore \boxed{\underset{z\to 1}{Lt}\ (z-1)F(z) = f(\infty)}$$

Problems:

① If $F(z) = \dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$ find $f(0)$ also find $\underset{t\to\infty}{Lt}\ f(t)$

Soln:

By initial value theorem,

$$f(0) = \underset{z\to\infty}{Lt}\ F(z)$$

$$= \underset{z\to\infty}{Lt}\ \frac{z(z-\cos aT)}{z^2-2z\cos aT+1} = \frac{\infty}{\infty}$$

$$= \underset{z\to\infty}{Lt}\ \frac{z(1)+(z-\cos aT)(1)}{2z-2\cos aT} \quad \text{by L'Hospital's rule}$$

$$= \underset{z\to\infty}{Lt}\ \frac{2z-\cos aT}{2z-2\cos aT} = \frac{\infty}{\infty}$$

$$= \underset{z\to\infty}{Lt}\ \frac{2}{2} = 1 \quad \text{by L'Hospital's rule.}$$

By Final value theorem,

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{z\to 1}{Lt}\ (z-1)F(z)$$

$$= \underset{z\to 1}{Lt}\ (z-1)\frac{z(z-\cos aT)}{z^2-2z\cos aT+1} = \frac{0}{0}$$

$$= \underset{z\to 1}{Lt}\ \frac{(z-1)[z(1)+(z-\cos aT)(1)]+z(z-\cos aT)(1)}{2z-2\cos aT}$$

$$\lim_{t\to\infty} f(t) = \frac{1-\cos aT}{2-2\cos aT} = \frac{1-\cos aT}{2(1-\cos aT)} = \frac{1}{2}$$

(2) If $U(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$, evaluate u_2 and u_3

Soln:

$$U(z) = \frac{2z^2+5z+14}{(z-1)^4}$$

$$= \frac{z^2\left[2+\frac{5}{z}+\frac{14}{z^2}\right]}{z^4\left[1-\frac{1}{z}\right]^4}$$

$$= \frac{1}{z^2}\,\frac{[2+5z^{-1}+14z^{-1}]}{[1-z^{-1}]^4}$$

By initial value theorem,

$$u_0 = \lim_{z\to\infty} U(z) = 0$$

$$u_1 = \lim_{z\to\infty}\left[z(U(z)-u_0)\right] = 0$$

$$u_2 = \lim_{z\to\infty}\left[z^2(U(z)-u_0-u_1 z^{-1})\right] = 2-0-0 = 2$$

$$u_3 = \lim_{z\to\infty} z^3(U(z)-u_0-u_1z^{-1}-u_2z^{-2})$$

$$= \lim_{z\to\infty} z^3\left[U(z)-0-0-2z^{-2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^2+5z+14}{(z-1)^4} - \frac{2}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{13z^3+2z^2+8z-2}{z^2(z-1)^4}\right]$$

$$u_3 = 13$$