Low pass Sampling

SAMPLING THEOREM: for low pass signals

A low pass or LP signal contains frequencies from 1Hz to some higher value.

Statement:

Sampling of the signal

"A band limited signal of finite energy, which has no frequency components higher than W hertz, is completely described by specifying the values of the signal at instants of time separated by $\frac{1}{2W}$ seconds &

Reconstruction of the signal

A band limited signal of finite energy, which has no frequency components higher than W hertz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second."

The above statement can be combined and stated alternately as follows.

"A continuous time signal can be completely represented in its samples and recovered back if the sampling frequency is twice of the highest frequency content of the signal i.e.,

$$\boxed{f_s \geq 2W} \rightarrow \text{Higher frequency content.}$$

Sampling frequency

PROOF: There are two parts.

1. Representation of $x(t)$ interms of its samples.
2. Reconstruction of $x(t)$ from its samples.

1. Representation of $x(t)$ in its Samples $x(nT_s)$

Step 1: Define $x_\delta(t)$

Step 2: Fourier transform of $x_\delta(t)$ i.e. $X_\delta(f)$

Step 3: Relation between $X(f)$ and $X_\delta(f)$

Step 4: Relation between $x(t)$ and $x(nT_s)$.

Step 1:

The sampled signal $x_\delta(t)$ is given as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s) \rightarrow (1)$$

$x(nT_s)$ is basically $x(t)$ sampled at $t = nT_s$,

$n = 0, \pm1, \pm2, \pm3, \ldots$

$x_\delta(t)$ is the product of $x(t)$ and impulse train $\delta(t)$.

$\delta(t-nT_s)$ indicates the samples placed at $\pm T_s$, $\pm 2T_s$, $\pm 3T_s$ ... and so on.

Step 2:

Taking Fourier transform of equation (1)

$$X_\delta(f) = FT\left\{\sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)\right\}$$

$$= FT\{\text{product of } x(t) \text{ and impulse train}\}$$

W.K.T

Fourier transform of product in time domain becomes convolution in frequency domain. i.e.,

$$X_\delta(f) = FT\{x(t)\} * FT\{\delta(t-nT_s)\} \longrightarrow (2)$$

By definitions,

$$x(t) \xleftrightarrow{FT} X(f) \text{ and}$$

$$\delta(t-nT_s) \xleftrightarrow{FT} f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

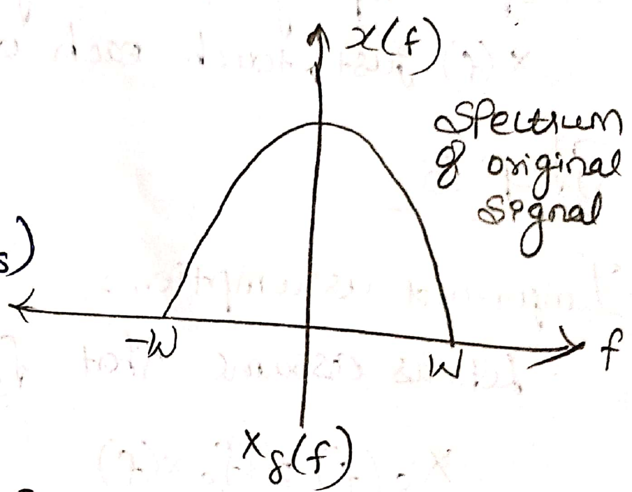


Fig: Spectrum of original signal X(f)

Hence equation (2) becomes

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Since convolution is linear,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f-nf_s)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f-nf_s) \quad \text{By shifting property of impulse function}$$

$$= \dots f_s X(f-2f_s) + f_s X(f-f_s) + f_s X(f) + f_s X(f-f_s) + f_s X(f-2f_s) + \dots$$

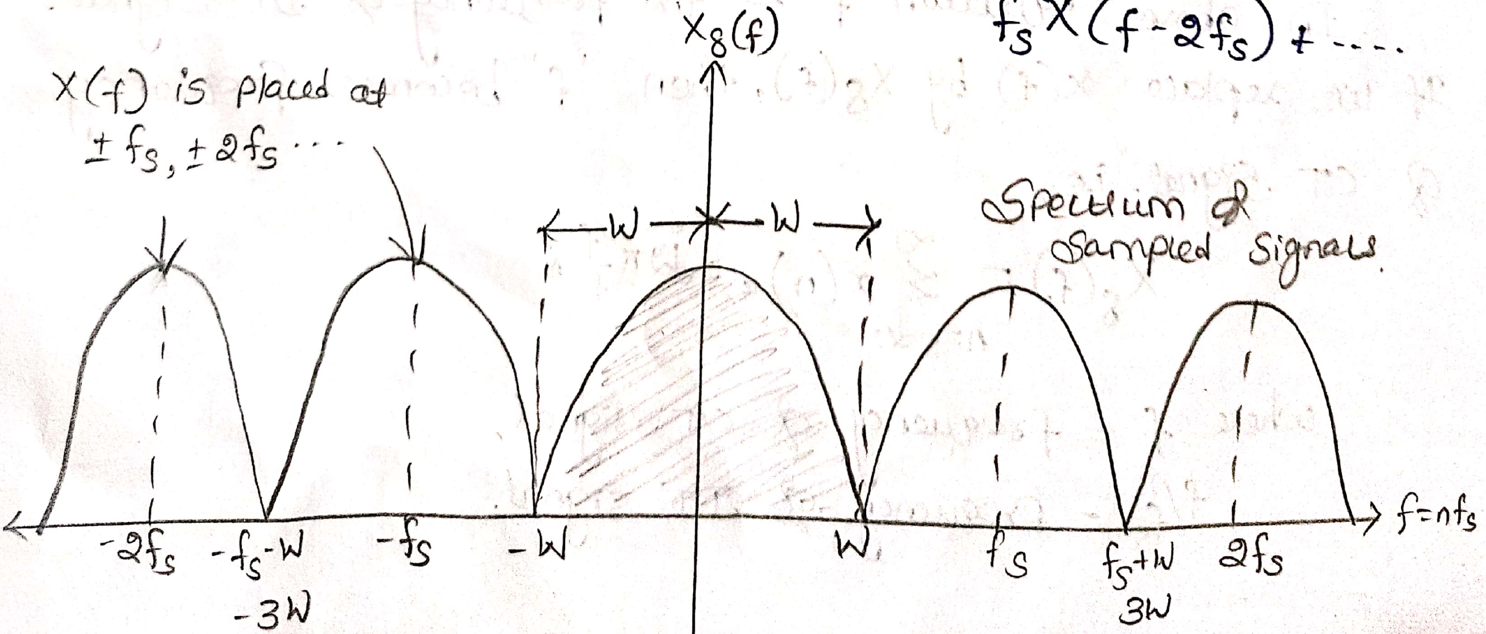


Fig: Spectrum sampled signal.

i) The R.H.S of above equation shows that $X(f)$ is placed at $\pm f_s, \pm 2f_s, \pm 3f_s \ldots$

ii) This means $X(f)$ is periodic in f_s.

iii) If sampling frequency is $f_s = 2W$, then the spectrum $X(f)$ just touch each other.

Step 3:

Important assumption:

Let us assume that $f_s = 2W$, then as per above diagram.

$$X_\delta(f) = f_s X(f) \quad \text{for } -W \leq f \leq W \ \& \ f_s = 2W$$

(or)

$$X(f) = \frac{1}{f_s} X_\delta(f) \quad \longrightarrow (3)$$

Step 4:

DTFT is, $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \quad \longrightarrow (4)$$

In above equation 'f' is the frequency of DT signal. If we replace $X(f)$ by $X_\delta(f)$, then 'f' becomes frequency of CT signal. i.e.,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \frac{f}{f_s} n}$$

where f = frequency of CT signal.

f/f_s = frequency of DT signal.

Since $x(n) = x(nT_s)$ i.e, Samples of $x(t)$, then we have

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \quad \text{since } \frac{1}{f_s} = T_s$$

Applying above expression in eq (3)

$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{j2\pi f n T_s}$$

Inverse Fourier Transform (IFT) of above equation gives $x(t)$ i.e.,

$$x(t) = \mathrm{IFT}\left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}\right\} \rightarrow (5)$$

→ Here $x(t)$ is represented completely interms of $x(nT_s)$.

→ Above equation holds for $f_s = 2W$. this means if the Samples are taken at the rate of $2W$ or higher, $x(t)$ is completely represented by its Samples.

→ First part of the Sampling theorem is proved by above two comments.

2. Reconstruction of $x(t)$ from its Samples.

Step 1: Take Inverse FT of $X(f)$ which is interms of $X_\delta(f)$.

Step 2: Show that $x(t)$ is obtained back with the help of interpolation function.

Step 1:

The IFT of equation (5) becomes

$$x(t) = \int_{-\infty}^{\infty} \left\{ \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \right\} e^{j2\pi f t} \, df$$

Here the integration can be taken from $-W \leq f \leq W$

Since $X(f) = \frac{1}{f_s} X_\delta(f)$ for $-W \leq f \leq W$.

$$x(t) = \int_{-W}^{W} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \cdot e^{j2\pi f t} \, df$$

Interchanging the order of summation & integration

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{f_s} \int_{-W}^{W} e^{j2\pi f (t-nT_s)} \, df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \cdot \left[\frac{e^{j2\pi f (t-nT_s)}}{j2\pi (t-nT_s)} \right]_{-W}^{W}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \left\{ \frac{e^{j2\pi W (t-nT_s)} - e^{-j2\pi W (t-nT_s)}}{j2\pi (t-nT_s)} \right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \cdot \frac{\sin 2\pi W (t-nT_s)}{\pi (t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi (2Wt - 2WnT_s)}{\pi (f_s t - f_s n T_s)}$$

Here $f_s = 2W$, hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$. Simplifying above equation

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi (2Wt - n)}{\pi (2Wt - n)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\, \text{sinc}(2Wt - n)$$

Since $\text{sinc}\,\theta = \frac{\sin \pi\theta}{\pi\theta}$

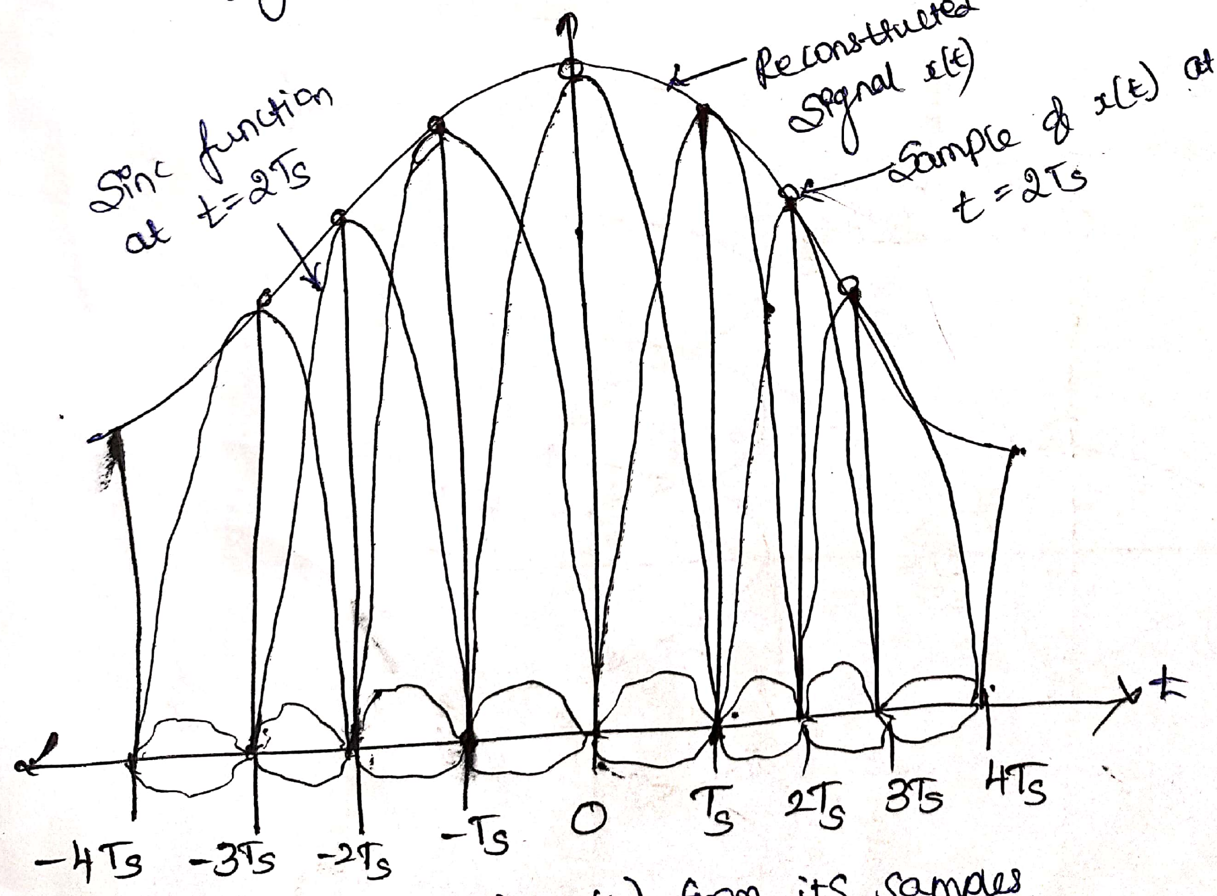
Step 2:

Let us interpret the above equation. Expanding we get,

$$x(t) = \dots + x(-2T_s)\,\text{sinc}(2Wt+2) + x(-T_s)\,\text{sinc}(2Wt+1) + x(0)\,\text{sinc}(2Wt) + x(T_s)\,\text{sinc}(2Wt-1) + \dots$$

1. The samples $x(nT_s)$ are weighted by sinc functions
2. The sinc function is the interpolating function.

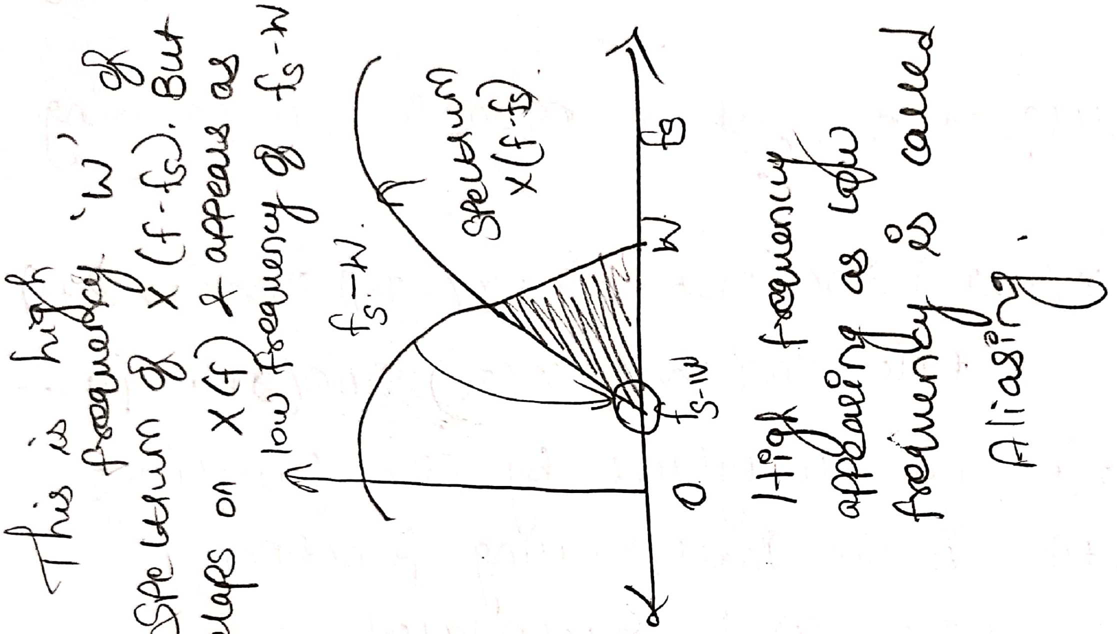
The fig. shows how $x(t)$ is interpolated.

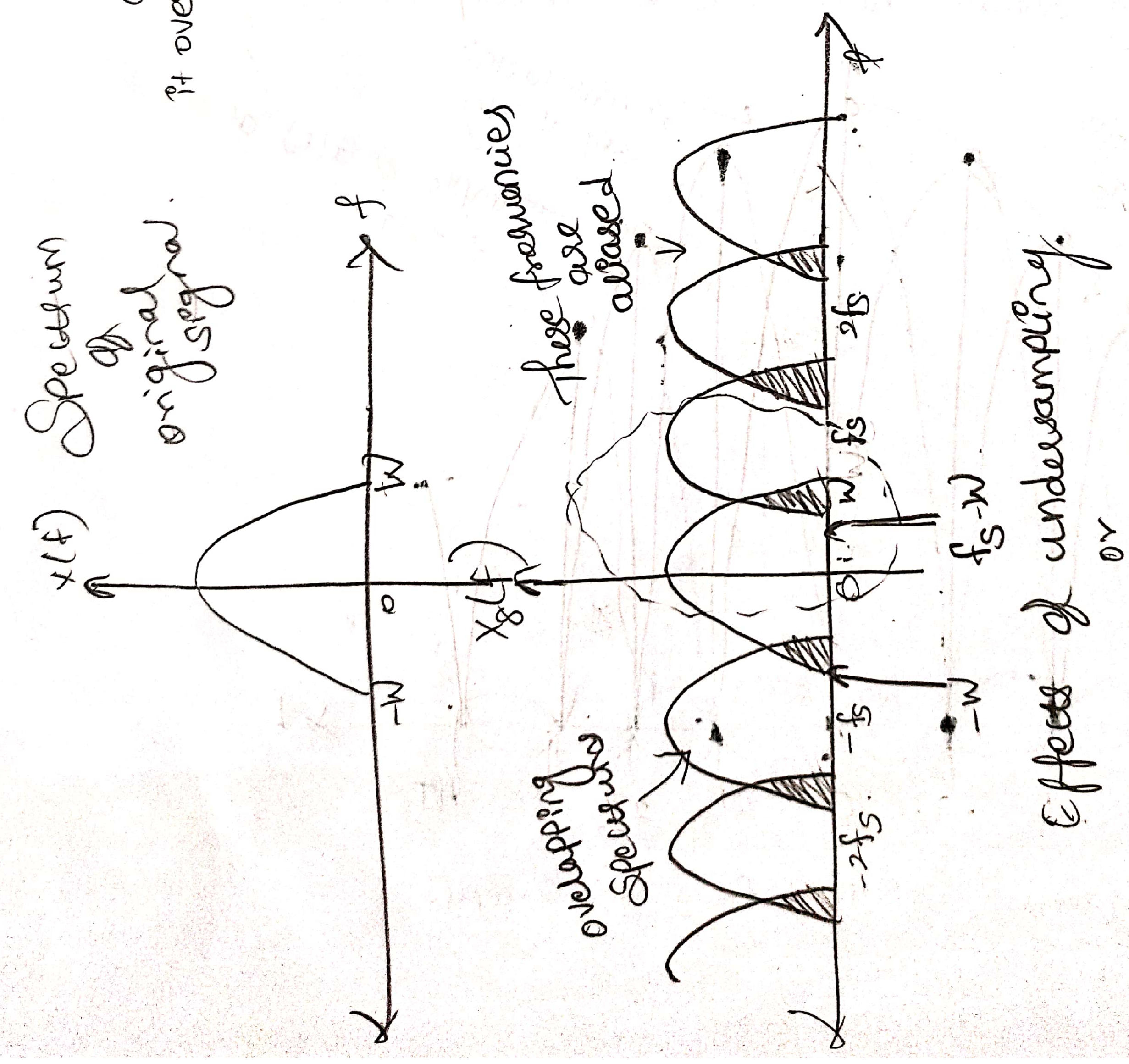


Reconstruction of $x(t)$ from its samples.

Effects of undersampling (Aliasing)

While proving sampling theorem we considered that $f_s = 2W$. Consider the case of $f_s < 2W$. Then the spectrum of $X_\delta(f)$ will be modified as follows.





Definition of Aliasing: "When the high frequency interferes with low frequency and appears as low frequency, then the phenomenon is called ALIASING".

Effects of Aliasing

1. Since high and low frequencies interfere with each other, distortion is generated.
2. The data is lost and it cannot be recovered.

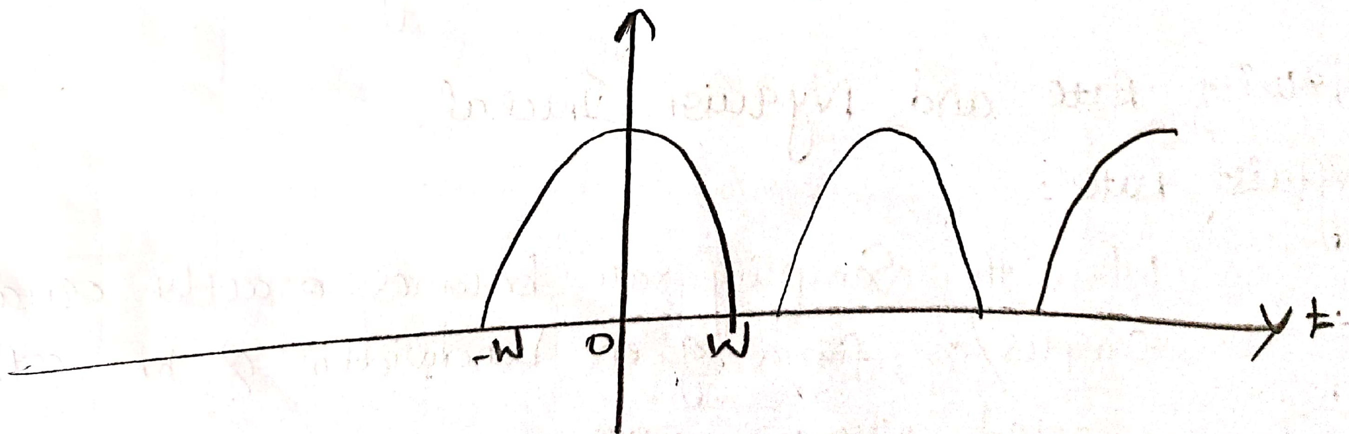
Different ways to Avoid Aliasing

Aliasing can be avoided by two methods.

1. Sampling rate $f_s \geq 2W$
2. Strictly bandlimit the signal to 'W'.

1. Sampling rate $f_s \geq 2W$

When the sampling rate is made higher than 2W, then the spectrums will not overlap and there will be sufficient gap between the individual spectrums.



Over Sampling:

When the signal is sampled at a rate much higher than Nyquist rate, it is called oversampling.

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It is necessary to avoid aliasing error in the signal. But it increases transmission bandwidth.

2. Bandlimiting the Signal.

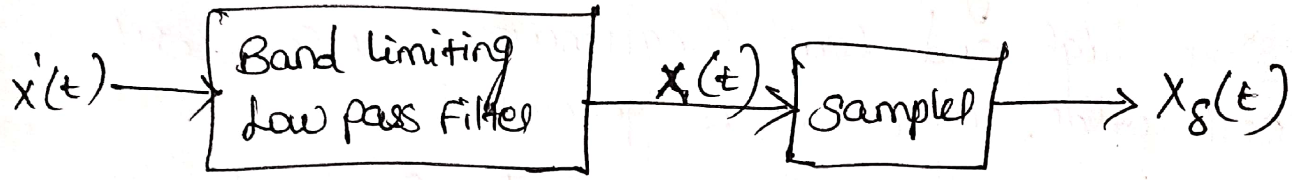

Fig: Bandlimiting the signal. The bandlimiting LPF is called PREALIAS FILTER.

The Sampling rate is $f_s = 2W$. Ideally speaking there should be no Aliasing. But there can be few components higher than $2W$. These components create Aliasing. Hence a low pass filter is used before sampling the signals.

→ Thus the o/p of low pass filter is strictly bandlimited & there are no frequency components higher than 'W'. Then there will be no aliasing.

Nyquist Rate and Nyquist Interval.

Nyquist Rate:

When the Sampling rate becomes exactly equal to '2W' samples/sec, for a given bandwidth of W Hertz, then it is called NYQUIST RATE.

$$\text{NYQUIST RATE} = 2W \text{ Hz}.$$

NYQUIST INTERVAL:

It is the time interval between any two adjacent samples when Sampling rate is nyquist Rate.

NYQUIST INTERVAL = $1/2W$ Seconds.