INFINITE PLATES

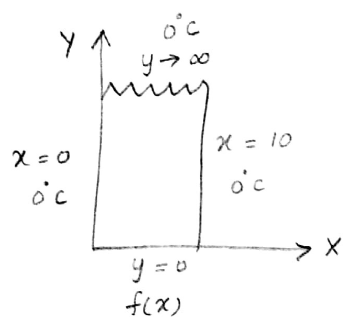
TYPE I : Vertically Infinite plates :

(1) A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that may be considered infinite in length without introducing appreciable error. The temperature at short edge $y = 0$ is given by

$$u = \begin{cases} 20x & \text{for } 0 \le x \le 5 \\ 20(10 - x) & \text{for } 5 \le x \le 10 \end{cases}$$

and the other three edges are kept at $0^\circ c$. Find the steady state temperature at any point in the plate.

Solution :



Step 1 :

The two-dimensional heat equation is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Step 2 :

The boundary conditions are,

(i) $u(0, y) = 0 \quad \forall\, y$

(ii) $u(10, y) = 0 \quad \forall\, y$

(iii) $u(x, \infty) = 0, \quad 0 < x < 10$

(iv) $u(x, 0) = \begin{cases} 20x & , 0 \le x \le 5 \\ 20(10 - x) & , 5 \le x \le 10 \end{cases}$

Step 3 :

The correct solution is,

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \longrightarrow (1)$$

Step 4 : Applying condition (i) in ①,

$$u(0,y) = c_1 (c_3 e^{py} + c_4 e^{-py})$$

$$0 = c_1 (c_3 e^{py} + c_4 e^{-py})$$

Here $c_3 e^{py} + c_4 e^{-py} \neq 0$ [$\because$ It is defined for all y]

$$\therefore \boxed{c_1 = 0}$$

$$① \Rightarrow u(x,y) = c_2 \sin px \left[c_3 e^{py} + c_4 e^{-py}\right] \rightarrow ②$$

Step 5 : Applying condition (ii) in ②,

$$u(10,y) = c_2 \sin(10p) \left[c_3 e^{py} + c_4 e^{-py}\right] = 0$$

Here $c_3 e^{py} + c_4 e^{-py} \neq 0$ [$\because$ It is defined for all y]

$c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$$\sin 10p = 0$$

$$\sin 10p = \sin n\pi$$

$$10p = n\pi$$

$$\boxed{p = \frac{n\pi}{10}}$$

$$② \Rightarrow u(x,y) = c_2 \sin\left(\frac{n\pi x}{10}\right) \left[c_3 e^{\frac{n\pi y}{10}} + c_4 e^{\frac{-n\pi y}{10}}\right] \rightarrow ③$$

Step 6 : Applying condition (iii) in ③,

$$u(x,\infty) = c_2 \sin\left(\frac{n\pi x}{10}\right) \left[c_3 e^{\infty} + c_4 e^{-\infty}\right] = 0$$

Here $c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$\sin\left(\frac{n\pi x}{10}\right) \neq 0$ [$\because$ It is defined for all x]

$$\therefore c_3 e^{\infty} + c_4 e^{-\infty} = 0$$

$$\Rightarrow c_3 e^{\infty} = 0 \quad (\because e^{-\infty} = 0)$$

$$\boxed{c_3 = 0} \quad (\because e^{\infty} \neq 0)$$

(Su)

$$③ \Rightarrow u(x,y) = c_2 \sin\left(\frac{n\pi x}{10}\right) c_4 e^{-n\pi y/10}$$

$$u(x,y) = c_n e^{-n\pi y/10} \sin\left(\frac{n\pi x}{10}\right)$$

where $c_n = c_2 c_4$.

Step 7: The most general solution is,

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right) e^{-n\pi y/10} \rightarrow ④$$

Step 8: Applying condition (iv) in ④,

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right) = f(x) \rightarrow ⑤$$

Step 9: To find c_n:

Expand $f(x)$ as a half range fourier sine series in (0, 10).

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) \rightarrow ⑥ \quad \text{where } b_n = \frac{2}{10}\int_0^{10} f(x) \sin\left(\frac{n\pi x}{10}\right) dx$$

From ⑤ & ⑥, $\boxed{b_n = c_n}$

$u = x,\ u' = 1$

$u = 10 - x,\ u' = -1$

$v = \sin\left(\frac{n\pi x}{10}\right)$

$v_1 = \dfrac{-\cos(n\pi x/10)}{n\pi/10}$

$v_2 = \dfrac{-\sin(n\pi x/10)}{(n\pi/10)^2}$

$$\therefore c_n = \frac{1}{5}\left\{ \int_0^5 20x \sin\left(\frac{n\pi x}{10}\right) dx + \int_5^{10} 20(10-x) \sin\left(\frac{n\pi x}{10}\right) dx \right\}$$

$$= \frac{1}{5}\left\{ \left[-x\left(\frac{10}{n\pi}\right)\cos\frac{n\pi x}{10} + \frac{\sin(n\pi x/10)}{(n\pi/10)^2} \right]_0^5 + \left[-(10-x)\frac{10}{n\pi}\cos\left(\frac{n\pi x}{10}\right) - \left(\frac{10}{n\pi}\right)^2 \sin\frac{n\pi x}{10} \right]_5^{10} \right\}$$

$$= \frac{1}{5}\left\{ -5 \times \frac{10}{n\pi}\cos\frac{n\pi}{2} + \left(\frac{10}{n\pi}\right)^2 \sin\frac{n\pi}{2} + 5 \times \frac{10}{n\pi}\cos\frac{n\pi}{2} + \left(\frac{10}{n\pi}\right)^2 \sin\frac{n\pi}{2} \right\}$$

$$C_n = 4 \times \frac{10^2}{n^2\pi^2}\, 2 \sin\left(\frac{n\pi}{2}\right)$$

$$\boxed{C_n = \frac{800}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)}$$

Step 10 : Subs the value of C_n in (4),

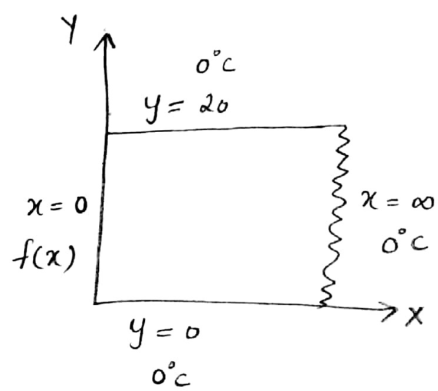
$$u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{20}\right) e^{-\frac{n\pi y}{10}}$$

(2) A rectangular plate with insulated surfaces is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $x=0$ is given by,

$$u = \begin{cases} 10y & , 0 \le y \le 10 \\ 10(20-y) & , 10 \le y \le 20 \end{cases}$$

and the two long edges as well as the other short edge are kept at $0^\circ c$. Find the steady state temperature distribution in the plate.

Solution :



Step 1 :

The two dimensional heat equation is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Step 2 :

The boundary conditions are

(i) $u(x,0) = 0 \quad \forall x$

(ii) $u(x,20) = 0 \quad \forall x$

(iii) $u(\infty,y) = 0 \quad , \; 0 < x < \infty$

(iv) $u(0,y) = \begin{cases} 10y & , 0 \le y \le 10 \\ 10(20-y) & , 10 \le y \le 20 \end{cases}$