

SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)



DEPARTMENT OF MATHEMATICS

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(8)

Problems on Vibrating String with Zero initial velocity:

orterded (1) A uniform String is Stretched and fastened to two Points x = 0 and x = 1 apart. Motion is started by displacing the string into the form of the curve y = kx(l-x)& then released from this position at time t=0. Derive

the expression for the displacement of any point on the string at a distance 'x' from one end at time t.

Solution:

Step 1: The wave equation is,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

B.c (i) : 4(0,t) = 0 for all t>0

B.c (ii): y(1,t) = 0 for all t>0

I.c (iii): $\left(\frac{\partial y}{\partial t}\right)_{(x_10)} = 0$, 0 < x < l (: initial velocity is Zero)

y(x,0) = Kx(l-x), 0 < x < l (initial displacement)

The suitable solution which satisfies our boundary Step 3: conditions are given by, -

 $y(x,t) = (c, cos(px + c_2 Sin px) (c_3 cos pat + c_4 Sin pat)$

Applying condition (i) in ear O, we get

y(0,t) = (c,+0)(c3 cospat + c4 sin pat) = 0.

Here (c3 cos pat + c4 Sin pat) + 0, since it is defined for all t >0.

subs c, = 0 in ean (), we get,



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$$y(x,t) = c_x \sin px (c_3 \cos pat + c_4 \sin pat) \rightarrow 2$$

Step 5:
Applying condition (ii) in ean (a), we get

$$y(l,t) = c_2 \sin pl \left(c_3 \cos pat + c_4 \sin pat\right) = 0$$
.

Here $(C_3 \cos pat + C_4 \sin pat) \neq 0$, since it is defined for all t > 0.

Therefore either ca = 0 or sin pl = 0.

If we take $c_a = 0$ we get a trivial solution.

.. Take sin pl = 0.

Sin pl = Sin nTT (: Sin nTT = 0)

Pl = nT

 $P = \frac{n\pi}{l}$, where n is an integer.

substituting $P = \frac{n\pi}{l}$ in earn (2), we get

 $y(x,t) = c_3 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l} \right)$ Step 6:

Diff 3 wire 't' pastially, we get,

$$\left(\frac{\partial y}{\partial t}\right)_{(x,t)} = c_{\lambda} \frac{\sin \frac{n\pi x}{l}}{l} \left(-c_{\beta} \left(\frac{n\pi a}{l}\right) \frac{\sin \frac{n\pi at}{l}}{l} + c_{4} \left(\frac{n\pi a}{l}\right)\right)$$

 $\cos \frac{n\pi at}{l}$

Now applying condition (iii), we get

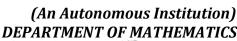
$$\left(\frac{\partial Y}{\partial t}\right)_{(\chi,0)} = \frac{c_2 \sin \frac{n\pi \chi}{\ell}}{\ell} \left(0 + \frac{c_4 \left(\frac{n\pi a}{\ell}\right)}{\ell}\right) = 0$$

$$\Rightarrow \frac{c_2 c_4 \left(\frac{n\pi a}{\ell}\right) \sin \frac{n\pi \chi}{\ell}}{\ell} = 0$$

Here $C_a \neq 0$, $Sin \frac{n\pi x}{l} \neq 0$: It is defined for all x and $n\pi a \neq 0$ since all are constants.



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Subs
$$C_4 = 0$$
 in ean 3, we get
$$y(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

$$y(x,t) = C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \rightarrow 4$$

where $c_n = c_2 c_3$.

Step 7: The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{\ell} \cos \frac{n\pi at}{\ell} \rightarrow 5$$

Step 8:

Applying B.c (iv) in earn (5), we have

$$y(x,0) = \frac{8}{5} C_n \sin \frac{n\pi x}{4} = kx(1-x) \rightarrow 6$$

$$\vdots \qquad \qquad n=1$$

Step 9:

To find Cn:

Expand Kx (1-x) in a half lange Fourier sine

series in the interval (0,1).

$$kx(1-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \rightarrow 7 \text{ where}$$

$$b_n = \frac{2}{2} \int f(x) \sin \frac{n\pi x}{2}$$

$$dx$$

$$C_n = \frac{2}{l} \int_{0}^{l} Kx (l-x) \sin \frac{n\pi x}{l} dx$$

$$C_{n} = \frac{2}{l} \begin{cases} -(klx - kx^{2}) \frac{1}{l} \cos \frac{n\pi x}{l} & u = klx - kx^{2} \\ -kl - 2kx & u' = kl - 2kx \\ +(kl - 2kx) \frac{l^{2}}{n^{2}\pi^{2}} \frac{\sin \frac{n\pi x}{l}}{l} & v = -\cos \frac{n\pi x}{l} \end{cases}$$

$$V = \sin \frac{n\pi x}{l}$$

$$V_{1} = -\cos \frac{n\pi x}{l}$$

$$- 2k \cdot \frac{l^3}{n^3 \pi^3} \cos \frac{n \pi x}{l} \int_0^l$$

$$C_{n} = \frac{2}{l} \left\{ -\frac{2k}{n^{3} \pi^{3}} \cos \frac{n\pi x}{l} \right\}_{0}^{l}$$

$$= -\frac{4kl^{2}}{n^{3} \pi^{3}} \left\{ \cos n\pi - \cos o \right\}$$

$$C_{n} = \frac{4kl^{2}}{n^{3} \pi^{3}} \left[1 - (-1)^{n} \right]$$

$$C_n = \int_{0}^{\infty} D$$
, if n is even
$$\left\{ \frac{8Kl^2}{n^3 \pi^3} \right\}$$
, if n is odd

Step10: subs the value of C_n in earn (5), we get

$$y(x,t) = \frac{8 k l^2}{n^3 \pi^3} \sin \frac{n \pi x}{l} \cos \frac{n \pi a t}{l}$$

$$y(x_{i}t) = \frac{8kl^{2}}{\pi^{3}} \sum_{n=odd}^{\infty} \frac{1}{n^{3}} \frac{\sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}}{l}$$

3 A tightly stretched string with fixed end points x=0 and x=1 is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{\Lambda}$. If it is deleased from rest from

this position find the displacement y at any distance X from one end at any time t.

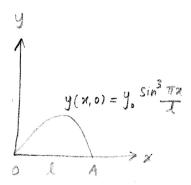
Solution:

Step1: The wave equation is

Stepa:
$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

(i)
$$y(0,t) = 0$$
 for all $t > 0$
(ii) $y(1,t) = 0$ for all $t > 0$



(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0$, 0 < x < l (initial velocity is Zero)

(iv)
$$y(x_{10}) = y_{0} \sin^{3} \frac{\pi x}{2}, 0 < x < 2$$
.

Step 3: The suitable solution which satisfies our boundary

y(x,t) = (c, cos px + c, sin px) (c, cos pat + c, sin pat) Conditions are given by,

Step4: Applying Condition (i) in ean (i), we get,

y(0,t) = (c, cos pxo)+ c2 sin pxo) (c3 cos pat + c4 sin pat) 'C, (c3 cos pat + c4 Sin pat) = 0

Here C3 cospat + C4 Sin pat + 0, since it is

Subs C1 = 0 in ean (1), we get, $y(x,t) = c_a \sin px (c_3 \cos pat + c_4 \sin pat)$ $\longrightarrow (2)$ Step 5:

Applying Condition (ii) in ean ②, we get,

y(1,t) = C2 Sin pl (C3 cos pat + C4 Sin pat) = 0.

Here (C3 cos pat + C4 sin pat) = 0, since it is

defined for all t >0.

Therefore either ca = 0 or Sin pl = 0

If we take $c_2 = 0$ we get a trivial solution.

Take Sin pl = 0.

Sin pl = Sin nT (: Sin nT = 0)

$$Pl = n\pi$$

$$P = n\pi$$
, where n is an integer.

subs p= nIT in ean a, we get,

$$y(x,t) = c_a \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right)$$
Step 6:

Diff 3 w.r.t 't' pastially, we get,

$$\left(\frac{\partial y}{\partial t}\right) = C_{a} \sin \frac{n\pi x}{l} \left(-C_{3}\left(\frac{n\pi a}{l}\right) \sin \frac{n\pi at}{l} + C_{4}\left(\frac{n\pi a}{l}\right)\right)$$

Cos mat Now applying condition (iii), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(\chi,0)} = c_a \sin \frac{n\pi x}{\ell} \left(0 + c_4 \left(\frac{n\pi a}{\ell}\right)\right) = \delta$$

$$\Rightarrow C_a C_4 \left(\frac{n\pi a}{l}\right) \sin \frac{n\pi x}{l} = 0$$

Here ca \$ 0, sin mx \$ 0 : it is defined for all 2

and nTTa to since all are constants.

subs c4 = 0 in earn 3, we get

$$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

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$$y(x_1t) = C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \rightarrow$$

where $c_n = c_2 c_3$

Step 7: The most general solution is,

The most general solution
$$y(x,t) = \frac{2}{n} C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \rightarrow 5$$

Step 8: Now applying condition (iv) in (5),

Now applying
$$y(x_1,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{1} = y_0 \sin \frac{3\pi x}{1}$$

$$y(x_1,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{1} = y_0 \sin \frac{3\pi x}{1} - \sin \frac{3\pi x}{1}$$

$$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{1} = \frac{y_0}{4} \left(3 \sin \frac{\pi x}{1} - \sin \frac{3\pi x}{1}\right)$$

$$\begin{bmatrix} \vdots & \sin^3 x & = \frac{1}{4} (3 \sin x) \\ -\sin 3x & \end{bmatrix}$$

$$\therefore c_1 \sin \frac{\pi x}{l} + c_2 \sin \frac{\pi x}{l} + c_3 \sin \frac{3\pi x}{l} + \cdots = \frac{y_o}{4} \left(\frac{3 \sin \frac{\pi x}{l}}{l} + \frac{y_o}{l} \right)$$

Equating like coefficients on either side, we have — sin 311x $C_1 = \frac{3y_0}{4}$, $C_2 = 0$, $C_3 = -\frac{y_0}{4}$, $C_4 = 0$, $C_5 = 0$,....

.: Ean (5) gives

$$y(x,t) = c_1 \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} + c_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi at}{l} + \frac{1}{2} \cos \frac{2\pi at}{l} + \frac{1$$

Step 9:

subs the values of c, c2, c3, ... in 6,

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l}$$

Note: Sin3 TX contains only two terms. Hence we need not expand $\sin^3 \frac{\pi x}{l}$ in a half range Sine series.