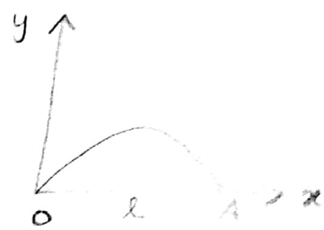
③ (8)

Problems on Vibrating String with Zero initial velocity :

extended attached

① A uniform string is stretched and fastened to two points $x = 0$ and $x = l$ apart. Motion is started by displacing the string into the form of the curve $y = kx(l-x)$ & then released from this position at time $t = 0$. Derive the expression for the displacement of any point on the string at a distance 'x' from one end at time t.

Solution :



Step 1: The wave equation is,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Step 2 :

B.C (i) : $y(0,t) = 0$ for all $t > 0$

B.C (ii) : $y(l,t) = 0$ for all $t > 0$

I.C (iii) : $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0 , \ 0 < x < l$ ($\because$ initial velocity is zero)

I.C (iv) : $y(x,0) = kx(l-x) , \ 0 < x < l$ (initial displacement)

Step 3:

The suitable solution which satisfies our boundary conditions are given by,

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \longrightarrow ①$$

Step 4 :

Applying condition (i) in eqn ①, we get

$$y(0,t) = (c_1 + 0)(c_3 \cos pat + c_4 \sin pat) = 0.$$

Here $(c_3 \cos pat + c_4 \sin pat) \neq 0$, since it is defined for all $t > 0$.

$$\therefore \boxed{c_1 = 0}$$

Subs $c_1 = 0$ in eqn ①, we get,

$$y(x,t) = c_2 \sin px \,(c_3 \cos pat + c_4 \sin pat) \longrightarrow (2)$$

Step 5 :

Applying condition (ii) in eqn (2), we get

$$y(l,t) = c_2 \sin pl \,(c_3 \cos pat + c_4 \sin pat) = 0.$$

Here $(c_3 \cos pat + c_4 \sin pat) \neq 0$, since it is defined for all $t > 0$.

Therefore either $c_2 = 0$ or $\sin pl = 0$.

If we take $c_2 = 0$ we get a trivial solution.

$\therefore$ Take $\sin pl = 0$.

$$\sin pl = \sin n\pi \quad (\because \sin n\pi = 0)$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}, \text{ where } n \text{ is an integer.}$$

Substituting $p = \frac{n\pi}{l}$ in eqn (2), we get

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l} \right) \longrightarrow (3)$$

Step 6 :

Diff (3) w.r.t 't' partially, we get,

$$\left(\frac{\partial y}{\partial t}\right)_{(x,t)} = c_2 \sin \frac{n\pi x}{l} \left(-c_3 \left(\frac{n\pi a}{l}\right) \sin \frac{n\pi at}{l} + c_4 \left(\frac{n\pi a}{l}\right) \cos \frac{n\pi at}{l} \right)$$

Now applying condition (iii), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = c_2 \sin \frac{n\pi x}{l} \left(0 + c_4 \left(\frac{n\pi a}{l}\right)\right) = 0$$

$$\Rightarrow c_2 c_4 \left(\frac{n\pi a}{l}\right) \sin \frac{n\pi x}{l} = 0.$$

Here $c_2 \neq 0$, $\sin \frac{n\pi x}{l} \neq 0$ $\because$ It is defined for all x and $\frac{n\pi a}{l} \neq 0$ since all are constants.

(4)

$\therefore \boxed{C_4 = 0}$

subs $C_4 = 0$ in eqn (3), we get

$$y(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

$$y(x,t) = C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \rightarrow (4)$$

where $C_n = C_2 C_3$.

Step 7: The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \rightarrow (5)$$

Step 8:

Applying B.C (iv) in eqn (5), we have

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = Kx(l-x) \rightarrow (6)$$

Step 9:

To find C_n:

Expand $Kx(l-x)$ in a half range Fourier sine series in the interval $(0,l)$.

$$Kx(l-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \rightarrow (7) \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

From (6) & (7) $b_n = C_n$

$$\therefore C_n = \frac{2}{l}\int_0^l Kx(l-x) \sin \frac{n\pi x}{l}\, dx$$

$$\therefore C_n = \frac{2}{l}\left\{-(Klx - Kx^2)\frac{l}{n\pi}\cos\frac{n\pi x}{l} + (Kl - 2Kx)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - 2K\cdot\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right\}_0^l$$

$u = Klx - Kx^2$

$u' = Kl - 2Kx$

$u'' = -2K$

$v = \sin \frac{n\pi x}{l}$

$v_1 = -\cos\frac{n\pi x}{l} \Big/ \frac{n\pi}{l}$

$v_2 = -\sin\frac{n\pi x}{l} \Big/ \frac{n^2\pi^2}{l^2}$

$v_3 = \cos\frac{n\pi x}{l} \Big/ \frac{n^3\pi^3}{l^3}$

$$C_n = \frac{2}{l}\left\{ -2k \frac{l^3}{n^3\pi^3} \cos\frac{n\pi x}{l} \right\}_0^l$$

$$= \frac{-4kl^2}{n^3\pi^3}\left\{ \cos n\pi - \cos 0 \right\}$$

$$C_n = \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\boxed{C_n = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \dfrac{8kl^2}{n^3\pi^3} & , \text{ if } n \text{ is odd} \end{cases}}$$

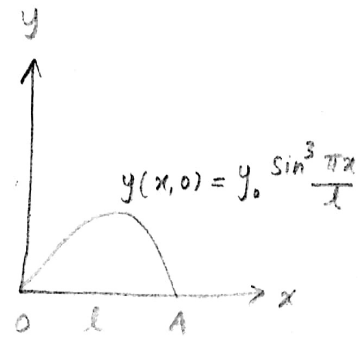
Step 10: subs the value of C_n in eqn 5, we get

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

$$y(x,t) = \frac{8kl^2}{\pi^3} \sum_{n=odd}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

3. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position find the displacement y at any distance x from one end at any time t.

Solution :



Step 1: The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Step 2:

The boundary conditions are

(i) $y(0,t) = 0$ for all $t > 0$

(ii) $y(l,t) = 0$ for all $t > 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0$, $0 < x < l$ (initial velocity is zero)

(iv) $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$, $0 < x < l$.

Step 3:

The suitable solution which satisfies our boundary Conditions are given by,

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \to (1)$$

Step 4: Applying condition (i) in eqn (1), we get,

$$y(0,t) = (c_1 \cos p(0) + c_2 \sin p(0))(c_3 \cos pat + c_4 \sin pat) = 0$$

$$c_1 (c_3 \cos pat + c_4 \sin pat) = 0$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$, since it is defined for all $t > 0$.

$$\therefore \boxed{c_1 = 0}$$

Subs $c_1 = 0$ in eqn (1), we get,

$$y(x,t) = c_2 \sin px \, (c_3 \cos pat + c_4 \sin pat) \quad \to (2)$$

Step 5 :

Applying condition (ii) in eqn ②, we get,

$$y(l,t) = c_2 \sin pl \,(c_3 \cos pat + c_4 \sin pat) = 0.$$

Here $(c_3 \cos pat + c_4 \sin pat) \neq 0$, since it is defined for all $t > 0$.

Therefore either $c_2 = 0$ or $\sin pl = 0$.

If we take $c_2 = 0$ we get a trivial solution.

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$$\boxed{p = \frac{n\pi}{l}}, \text{ where } n \text{ is an integer.}$$

Subs $p = \frac{n\pi}{l}$ in eqn ②, we get,

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \longrightarrow ③$$

Step 6 :

Diff ③ w.r.t 't' partially, we get,

$$\left(\frac{\partial y}{\partial t}\right) = c_2 \sin \frac{n\pi x}{l}\left(-c_3\left(\frac{n\pi a}{l}\right)\sin \frac{n\pi at}{l} + c_4\left(\frac{n\pi a}{l}\right)\cos \frac{n\pi at}{l}\right)$$

Now applying condition (iii), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = c_2 \sin \frac{n\pi x}{l}\left(0 + c_4\left(\frac{n\pi a}{l}\right)\right) = 0$$

$$\Rightarrow c_2 c_4 \left(\frac{n\pi a}{l}\right)\sin \frac{n\pi x}{l} = 0$$

Here $c_2 \neq 0$, $\sin \frac{n\pi x}{l} \neq 0$ $\because$ it is defined for all x and $\frac{n\pi a}{l} \neq 0$ since all are constants.

$$\therefore \boxed{c_4 = 0}$$

Subs $c_4 = 0$ in eqn ③, we get

$$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

$$y(x,t) = c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \rightarrow (4)$$

where $c_n = c_2 c_3$.

Step 7:

The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \rightarrow (5)$$

Step 8:

Now applying condition (iv) in (5),

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = y_0 \sin^3 \frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = \frac{y_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right)$$

$$\left[\because \sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)\right]$$

$$\therefore c_1 \sin \frac{\pi x}{l} + c_2 \sin \frac{2\pi x}{l} + c_3 \sin \frac{3\pi x}{l} + \cdots = \frac{y_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right)$$

Equating like coefficients on either side, we have

$$c_1 = \frac{3y_0}{4}, \quad c_2 = 0, \quad c_3 = -\frac{y_0}{4}, \quad c_4 = 0, \quad c_5 = 0, \ldots$$

$\therefore$ Eqn (5) gives

$$y(x,t) = c_1 \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} + c_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi a t}{l} + c_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l} + \cdots \rightarrow (6)$$

Step 9:

Subs the values of $c_1, c_2, c_3, \ldots$ in (6),

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l}$$

Note: $\sin^3 \frac{\pi x}{l}$ contains only two terms. Hence we need not expand $\sin^3 \frac{\pi x}{l}$ in a half range sine series.