



DEPARTMENT OF MATHEMATICS

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TYPE 3 : Rectangular plate : Temperature given in horizontal edge:

- (1) Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions $u(0, y) = u(a, y) = 0$ for $0 \leq y \leq b$, $u(x, b) = 0$ and $u(x, 0) = x(a-x)$ for $0 \leq x \leq a$.

Solution:

Step 1: The steady state two dimension equation is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Step 2: The boundary conditions are

- (i) $u(0, y) = 0$
- (ii) $u(a, y) = 0$
- (iii) $u(x, b) = 0$
- (iv) $u(x, 0) = x(a-x)$

Step 3: The suitable solution is given by,

$$u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py}) \rightarrow (1)$$

Step 4: Applying condition (i) in (1),

$$u(0, y) = c_1 (c_3 e^{py} + c_4 e^{-py}) = 0$$

Here $c_3 e^{py} + c_4 e^{-py} \neq 0$ [\because It is defined for all y]

$$\therefore c_1 = 0$$

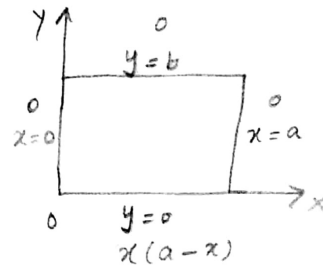
Subs $c_1 = 0$ in (1);

$$u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \rightarrow (2)$$

Step 5: Applying condition (ii) in (2),

$$u(a, y) = c_2 \sin pa (c_3 e^{py} + c_4 e^{-py}) = 0$$

Here $c_3 e^{py} + c_4 e^{-py} \neq 0$ [\because It is defined for all y]





$$c_2 \neq 0 \quad [\because c_1 = 0 \text{ we get a trivial solution}]$$

$$\therefore \sin p a = 0 = \sin n \pi$$

$$p a = n \pi$$

$$\boxed{p = \frac{n \pi}{a}}$$

subs $p = \frac{n \pi}{a}$ in (2),

$$u(x, y) = c_2 \sin \frac{n \pi x}{a} (c_3 e^{n \pi y/a} + c_4 e^{-n \pi y/a}) \rightarrow (3)$$

Step b: Applying Condition (ii) in (3),

$$u(x, b) = c_2 \sin \left(\frac{n \pi x}{a} \right) [c_3 e^{n \pi b/a} + c_4 e^{-n \pi b/a}]$$

Here $c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$$\sin \left(\frac{n \pi x}{a} \right) \neq 0 \quad [\because \text{It is defined for all } x]$$

$$c_3 e^{n \pi b/a} + c_4 e^{-n \pi b/a} = 0$$

$$c_3 e^{n \pi b/a} = -c_4 e^{-n \pi b/a} = \frac{-c_4}{e^{n \pi b/a}}$$

$$\boxed{c_4 = -c_3 e^{2n \pi b/a}}$$

subs the value of c_4 in (3),

$$u(x, y) = c_2 \sin \left(\frac{n \pi x}{a} \right) [c_3 e^{n \pi y/a} - c_3 e^{-n \pi y/a} e^{2n \pi b/a}]$$

$$= c_2 c_3 \sin \left(\frac{n \pi x}{a} \right) \left[e^{n \pi y/a} - \frac{e^{n \pi b/a}}{e^{-n \pi b/a}} \cdot e^{-n \pi y/a} \right]$$

$$= c_0 \sin \left(\frac{n \pi x}{a} \right) \left[\frac{e^{-\frac{n \pi}{a}(b-y)} - e^{\frac{n \pi}{a}(b-y)}}{e^{-n \pi b/a}} \right]$$

$$= -c_0 e^{n \pi b/a} \sin \left(\frac{n \pi x}{a} \right) \left[e^{+\frac{n \pi}{a}(b-y)} - e^{-\frac{n \pi}{a}(b-y)} \right]$$

$$= -2c_0 e^{n \pi b/a} \sin \left(\frac{n \pi x}{a} \right) \sinh (b-y) \frac{n \pi}{a}$$



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Step 7: The most general solution is ,

$$u(x, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh(b-y) \frac{n\pi}{a} \rightarrow (4)$$

where $C_n = -2c_0 e^{n\pi b/a}$

Step 8: Applying condition (iv) in (4), we get

$$\begin{aligned} u(x, 0) &= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) = x(a-x) \\ &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) = x(a-x) \rightarrow (5) \end{aligned}$$

where $B_n = C_n \sinh\left(\frac{n\pi b}{a}\right)$

Expand $f(x)$ as a half range Fourier sine series

in $(0, a)$.

$$x(a-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \rightarrow (6)$$

where $b_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$

From (5) & (6), $B_n = b_n$

$$B_n = \frac{2}{a} \int_0^a x(a-x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a} \left\{ -x(a-x) \frac{a}{n\pi} \cos \frac{n\pi x}{a} + \frac{(a-2x)a^2}{n^2\pi^2} \sin \frac{n\pi x}{a} - \frac{2a^3}{n^3\pi^3} \cos \frac{n\pi x}{a} \right\}_0^a$$

$$= \frac{2}{a} \left\{ -a(a-a) \frac{a}{n\pi} \cos n\pi - \frac{2a^3}{n^3\pi^3} \cos n\pi + \frac{2a^3}{n^3\pi^3} \right\}$$

$$u = x(a-x)$$

$$u' = a-2x$$

$$u'' = -2$$

$$V = \sin \frac{n\pi x}{a}$$

$$V_1 = -\frac{\cos \frac{n\pi x}{a}}{n\pi a}$$

$$V_2 = -\frac{\sin \frac{n\pi x}{a}}{n^2\pi^2/a}$$

$$V_3 = \frac{\cos \frac{n\pi x}{a}}{n^3\pi^3/a^3}$$

$$B_n = \frac{2}{a} \cdot \frac{2a^3}{n^3 \pi^3} [1 - (-1)^n]$$

$$B_n = \frac{4a^2}{n^3 \pi^3} [1 - (-1)^n]$$

$$B_n = \begin{cases} \frac{8a^2}{n^3 \pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Step 9: Subs the value of B_n in (4),

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4a^2 [1 - (-1)^n]}{n^3 \pi^3 \sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right)$$