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TYPE I : SQUARE PLATES : Temperature given in horizontal edge :

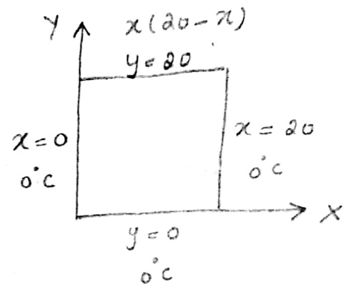
(1) 1 A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$, $0 < x < 20$ while the other ~~two~~ three edges are kept at $0^\circ c$. Find the steady state temperature distribution in the plate.

Solution :

Step 1 :

The two dimensional heat equation is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



Step 2 :

The boundary conditions are,

(i) $u(0, y) = 0$, $0 < y < 20$

(ii) $u(20, y) = 0$, $0 < y < 20$

(iii) $u(x, 0) = 0$, $0 < x < 20$

(iv) $u(x, 20) = x(20 - x)$, $0 < x < 20$

Step 3 :

The correct solution is,

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \longrightarrow (1)$$

Step 4 : Applying condition (i) in (1),

$$u(0, y) = c_1 (c_3 e^{py} + c_4 e^{-py}) = 0$$

Here $c_3 e^{py} + c_4 e^{-py} \neq 0$ [$\because$ It is defined for all y]

$$\therefore \boxed{c_1 = 0}$$

$$(1) \Rightarrow u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \longrightarrow (2)$$

Step 5 : Applying condition (ii) in ②,

$$u(20, y) = c_2 \sin 20p \, (c_3 e^{py} + c_4 e^{-py}) = 0$$

Here $c_3 e^{py} + c_4 e^{-py} \neq 0$ [$\because$ It is defined for all y]

$c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$$\therefore \sin 20p = 0$$

$$\sin 20p = \sin n\pi$$

$$20p = n\pi$$

$$\boxed{p = \frac{n\pi}{20}}$$

Subs the value of p in ②,

$$u(x, y) = c_2 \sin\left(\frac{n\pi x}{20}\right)\left[c_3 e^{\frac{n\pi y}{20}} + c_4 e^{-\frac{n\pi y}{20}}\right] \rightarrow ③$$

Step 6 :

Applying condition (iii) in ③,

$$u(x, 0) = c_2 \sin\left(\frac{n\pi x}{20}\right)(c_3 + c_4) = 0$$

Here $c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$\sin\left(\frac{n\pi x}{20}\right) \neq 0$ [$\because$ It is defined for all x]

$$\therefore c_3 + c_4 = 0$$

$$\boxed{c_4 = -c_3}$$

$$③ \Rightarrow u(x, y) = c_2 \sin\left(\frac{n\pi x}{20}\right)\left[c_3 e^{\frac{n\pi y}{20}} - c_3 e^{-\frac{n\pi y}{20}}\right]$$

$$= c_2 c_3 \sin\left(\frac{n\pi x}{20}\right)\left[e^{n\pi y/20} - e^{-n\pi y/20}\right]$$

$$= c_2 c_3 \sin\left(\frac{n\pi x}{20}\right)\left[2 \sinh\left(\frac{n\pi y}{20}\right)\right]$$

$$\therefore u(x, y) = c_n \sin\left(\frac{n\pi x}{20}\right) \sinh\left(\frac{n\pi y}{20}\right)$$ where $c_n = 2c_2 c_3$

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Step 7 : The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{20}\right) \sinh\left(\frac{n\pi y}{20}\right) \longrightarrow (4)$$

Step 8 : Applying condition (iv) in (4),

$$u(x,20) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{20}\right) \sinh(n\pi) = x(20-x) \longrightarrow (5)$$

Step 9 : To find c_n :

Expand $f(x) = x(20-x)$ as a half range Fourier sine series in $(0,20)$.

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{20}\right) \longrightarrow (6)$$

where $b_n = \frac{2}{20}\int_0^{20} f(x) \sin\left(\frac{n\pi x}{20}\right) dx$

From (5) & (6), $b_n = c_n \sinh(n\pi)$

$$\therefore c_n = \frac{1}{\sinh(n\pi)} \cdot \frac{1}{10} \int_0^{20} (20x - x^2) \sin \frac{n\pi x}{20}\, dx$$

$$= \frac{1}{10 \sinh(n\pi)} \left\{ (20x-x^2)\left(\frac{-\cos\frac{n\pi x}{20}}{n\pi/20}\right) + (20-2x)\frac{\sin\frac{n\pi x}{20}}{(n\pi/20)^2} - 2\,\frac{\cos\frac{n\pi x}{20}}{(n\pi/20)^3} \right\}_0^{20}$$

$$= \frac{1}{10 \sinh n\pi}\left(\frac{-2 \times 20^3}{n^3 \pi^3}\right)\left[\cos n\pi - 1\right]$$

$$\boxed{c_n = \frac{800}{\sinh n\pi} \times \frac{1}{n^3\pi^3}\left[1-(-1)^n\right]}$$

$u = 20x - x^2$
$u' = 20 - 2x$
$u'' = -2$
$v = \sin n\pi x/20$
$v_1 = \frac{-\cos n\pi x/20}{n\pi/20}$
$v_2 = \frac{-\sin n\pi x/20}{(n\pi/20)^2}$
$v_3 = \frac{\cos n\pi x/20}{(n\pi/20)^3}$

Step 10 : Subs the value of c_n in (4),

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800\left[1-(-1)^n\right]}{n^3\pi^3 \sinh(n\pi)} \sin\left(\frac{n\pi x}{20}\right) \sinh\left(\frac{n\pi y}{20}\right)$$