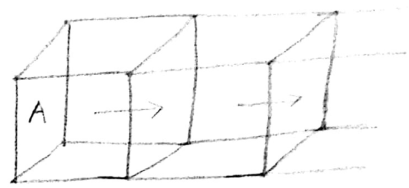
THE ONE-DIMENSIONAL HEAT EQUATION

Derivation of one-dimensional heat equation:

Consider a long thin bar (or wire (or) rod) of constant cross sectional area A and homogeneous conducting material. Let ρ be the density of the material, c be a specific heat and k be the thermal conductivity of the material. We assume that the surface of the bar is insulated so that the heat flow along parallel lines which are perpendicular to the area A.



One dimensional heat equation:

The one-dimensional heat flow equation is

$$\boxed{\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}}$$

Where $\alpha^2 = \dfrac{k}{\rho c} = \dfrac{\text{Thermal conductivity}}{\text{Density} \times \text{Specific heat}}$

Various Solutions of one-dimensional heat equation:

The various solutions of one-dimensional heat equation is

(i) $u(x,t) = (c_1 e^{px} + c_2 e^{-px})\, e^{-\alpha^2 p^2 t}$

(ii) $u(x,t) = (c_1 \cos px + c_2 \sin px)\, e^{-\alpha^2 p^2 t}$

(iii) $u(x,t) = (c_1 x + c_2)\, c_3$

The most suitable solution is

$u(x,t) = (c_1 \cos px + c_2 \sin px)\, e^{-\alpha^2 p^2 t}$

(25)

Type I : Problems with Zero Boundary Values [i.e., the temperatures at the ends of the rod are kept at zero]

① A rod of length 'l' with insulated sides is initially at a uniform temperature $f(x) = K(lx - x^2)$, $0 < x < l$. Its ends are suddenly cooled to $0°c$ and are kept at that temperature. Find the temperature function $u(x,t)$

(or)

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(0,t) = 0$, $u(l,t) = 0$ & $u(x,0) = K(lx - x^2)$, $0 < x < l$

Solution :

Step 1 : The one dimensional heat equation is given by,

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Step 2 : The boundary conditions are,

(i) $u(0,t) = 0 \quad \forall\, t$

(ii) $u(l,t) = 0 \quad \forall\, t$

(iii) $u(x,0) = K(lx - x^2)$, $0 < x < l$

Step 3 : The most suitable solution is given by,

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, e^{-\alpha^2 p^2 t} \rightarrow ①$$

Step 4 : Applying condition (i) in ①,

$$u(0,t) = c_1 e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [$\because$ It is defined for all t]

$$\therefore \boxed{c_1 = 0}$$

subs $c_1 = 0$ in ①, we get

$$u(x,t) = c_2 \sin px\, e^{-\alpha^2 p^2 t} \rightarrow ②$$

Step 5: Applying condition (ii) in ②,

$$u(l,t) = c_2 \sin pl \, e^{-\alpha^2 p^2 t} = 0$$

$e^{-\alpha^2 p^2 t} \neq 0$ [$\because$ It is defined for all t]

$c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$\therefore \sin pl = 0$

$\sin pl = \sin n\pi$ [$\because \sin n\pi = 0$]

$pl = n\pi$

$$\boxed{p = \frac{n\pi}{l}}$$

Subs the value of p in ②,

$$u(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t / l^2}$$

Step 6:

The most general solution is,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \rightarrow ③$$

Step 7: Applying condition (iii) in ③,

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = k(lx - x^2) \rightarrow ④$$

Step 8: To find c_n:

Expand $f(x) = k(lx - x^2)$ as a half range sine series in $(0, l)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \rightarrow ⑤$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$

From ④ & ⑤, $c_n = b_n$

$$\therefore c_n = \frac{2}{l} \int_0^l k(lx - x^2) \sin \frac{n\pi x}{l} \, dx$$

(2b)

$$C_n = \frac{2K}{l}\left\{ (lx - x^2)\left(-\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l} + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right\}_0^l$$

$u = lx - x^2$

$u' = l - 2x$, $u'' = -2$

$v = \sin n\pi x/l$

$v_1 = -\cos\frac{n\pi x}{l} \Big/ \frac{n\pi}{l}$

$v_2 = -\sin\frac{n\pi x}{l} \Big/ \frac{n^2\pi^2}{l^2}$

$v_3 = \cos\frac{n\pi x}{l} \Big/ \frac{n^3\pi^3}{l^3}$

$$= -\frac{2K}{l}\left(\frac{2l^3}{n^3\pi^3}\right)\left[\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= -\frac{4Kl^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$\boxed{C_n = \frac{4Kl^2}{n^3\pi^3}\left[1-(-1)^n\right]}$$

Step 9 : subs the value of C_n in (3),

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4Kl^2}{n^3\pi^3}\left[1-(-1)^n\right]\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

Type 2 : Steady State Conditions and Zero Boundary Conditions :

Steady State : The temperature does not vary w.r.t time 't' is called steady state.

Therefore, when steady state condition exists $u(x,t)$ becomes $u(x)$.

Steady State solution of one dimensional heat equation:

In unsteady state, one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

In steady state, $\frac{\partial u}{\partial t} = 0$

$$\therefore \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

$\therefore$ The general solution is $u(x) = ax + b$ where a and b are arbitrary constants.

Problems :

1. A rod 30 cm long has its end A and B kept at 20°c and 80° c respectively until steady state conditions prevails. Find the steady state temperature in the rod.

Solution:

The steady state one dimensional heat equation is,

$$\frac{d^2 u}{dx^2} = 0$$

The general solution is,

$$u(x) = ax + b \longrightarrow (1)$$

The boundary conditions are,

(i) $u(0) = 20^\circ c$

(ii) $u(30) = 80^\circ c$

Applying Condition (i) in ①,

$u(0) = b$

$\boxed{b = 20}$

Applying Condition (ii) in ①,

$u(30) = 30a + b$

$80 = 30a + 20$

$60 = 30a$

$\boxed{a = 2}$

$\therefore$ The solution is, $\boxed{u(x) = 2x + 20}$

② The ends A and B of a rod of length 10 cm long have their temperature kept 20°c and 70°c. Find the steady state temperature distribution on the rod.

Solution:

The steady state one dimensional heat equation is,

$$\frac{d^2u}{dx^2} = 0$$

The general solution is

$u(x) = ax + b \longrightarrow$ ①

The boundary conditions are,

(i) $u(0) = 20^\circ c$

(ii) $u(10) = 70^\circ c$

Applying Condition (i) in ①,

$u(0) = a(0) + b = 20$

$\boxed{b = 20}$

Applying condition (ii) in ①,

$u(10) = a(10) + b = 70$

$10a + 20 = 70$

$\boxed{a = 5}$

Subs a & b in ①,

$\boxed{u = 5x + 20}$