Type - II Clairaut's Form

A PDE is of the form $z = px + qy + f(p,q)$ is said to be of clairauts' form.

Here the complete integral of a PDE in clairauts' form is $z = ax + by + f(a,b)$

① Solve: $z = px + qy + p^2 + pq + q^2$

It is of the form $z = px + qy + f(p,q)$

The complete integral soln. is $z = ax + by + a^2 + ab + b^2$ — ① [put $p = a$ & $q = b$]

To find Singular Integral:

Diff ① w.r.to a

$$\frac{\partial z}{\partial a} = x + 2a + b \Rightarrow 0 = x + 2a + b$$

$$\Rightarrow x + 2a + b = 0$$

$$\Rightarrow a = -\frac{(x+b)}{2} \quad \text{— ②}$$

Diff ① w.r.to b.

$$\frac{\partial z}{\partial b} = y + a + 2b \Rightarrow 0 = y + a + 2b$$

$$\Rightarrow b = -\frac{(y+a)}{2} \quad \text{— ③}$$

Sub ③ in ②.

$$a = -\frac{1}{2}\left[x - \frac{(y+a)}{2}\right] = -\frac{1}{2}\left[\frac{2x - y \cdot a}{2}\right] = \frac{-2x + y + a}{4}$$

$$4a = -2x + y + a$$

$$a = -\frac{2x + y}{3}$$

Sub ② in ③

$$b = \frac{-y-a}{2} = \frac{1}{2}\left[y - \left(\frac{-2x+y}{3}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{3y+2x-y}{3}\right]$$

$$b = \frac{2x-4y}{6}$$

$$\Rightarrow b = \frac{x-2y}{3}.$$

$\therefore$ Singular Integral soln is

$$z = \left(\frac{y-2x}{3}\right)x + \left(\frac{x-2y}{3}\right)y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

② Solve: $z = px + qy + 2\sqrt{pq}$.

It is of the form $z = px + qy + f(p,q)$

Complete Integral soln is

$$\Rightarrow z = ax + by + 2\sqrt{ab}.$$

To find singular Integral:

D.w.r.to a.

$$\frac{\partial z}{\partial a} = x + 2\cdot\frac{1}{2\sqrt{ab}}(b)$$

$$0 = x + \frac{b}{\sqrt{ab}}$$

$$\Rightarrow 0 = x + \frac{\sqrt{b}}{\sqrt{a}} \Rightarrow x = -\frac{\sqrt{b}}{\sqrt{a}}.$$

$$\Rightarrow \sqrt{a} = -\frac{\sqrt{b}}{x} \quad \text{—— ①}$$

D.w.r.to b

$$\frac{\partial z}{\partial b} = y + 2 \cdot \frac{1}{2\sqrt{ab}}(a) = y + \frac{a}{\sqrt{ab}}$$

$$\Rightarrow 0 = y + \frac{\sqrt{a}}{\sqrt{b}} \Rightarrow y = -\frac{\sqrt{a}}{\sqrt{b}}$$

$$\Rightarrow \sqrt{b} = -\frac{\sqrt{a}}{y} \quad \text{--- (2)}$$

Sub (1) in (2). $\sqrt{b} = +\frac{\sqrt{b}}{xy} \Rightarrow 1 = \frac{1}{xy}$

$$\Rightarrow xy = 1$$

∴ Singular Integral soln is $xy = 1$

(3) Solve: $z = px + qy + \sqrt{p^2+q^2+1}$.

It is of the form $z = px + qy + f(p,q)$

Complete Integral soln is $z = ax + by + \sqrt{a^2+b^2+1}$

To find Singular Integral:

D.w.r.to a, $\frac{\partial z}{\partial a} = x + \frac{1}{2\sqrt{a^2+b^2+1}} \cdot 2a$

$$0 = x + \frac{a}{\sqrt{a^2+b^2+1}}$$

$$\Rightarrow x = -\frac{a}{\sqrt{a^2+b^2+1}} \Rightarrow a = -x\sqrt{a^2+b^2+1}$$

D.w.r.to b. $\frac{\partial z}{\partial b} = y + \frac{2b}{2\sqrt{a^2+b^2+1}}$

$$0 = y + \frac{b}{\sqrt{a^2+b^2+1}}$$

$$\Rightarrow b = -y\sqrt{a^2+b^2+1} \quad \text{---} (2)$$

In (1) & (2) taking square on both sides, we get

$$x^2 = \frac{a^2}{a^2+b^2+1} \quad \& \quad y^2 = \frac{b^2}{a^2+b^2+1}$$

$$\Rightarrow x^2+y^2 = \frac{a^2}{a^2+b^2+1} + \frac{b^2}{a^2+b^2+1} = \frac{a^2+b^2}{a^2+b^2+1}$$

$$\Rightarrow 1-(x^2+y^2) = 1-\left(\frac{a^2+b^2}{a^2+b^2+1}\right)$$

$$\Rightarrow a^2+b^2+1 = \frac{1}{1-x^2-y^2}.$$

Taking square OBS.

$$\sqrt{a^2+b^2+1} = \frac{1}{\sqrt{1-x^2-y^2}}$$

From (1), $a = -\frac{x}{\sqrt{a^2+b^2+1}} = -\frac{x}{\sqrt{1-x^2-y^2}}$

$$b = -\frac{y}{\sqrt{a^2+b^2+1}} = -\frac{y}{\sqrt{1-x^2-y^2}}.$$

Sub (a) & (b) in $z = ax + by + \sqrt{a^2+b^2+1}$

$$z = -\frac{x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$\Rightarrow \quad z\sqrt{1-x^2-y^2} = 1-x^2-y^2$

$\Rightarrow \quad z^2(1-x^2-y^2) = (1-x^2-y^2)^2$

$\Rightarrow \quad x^2+y^2+z^2 = 1$