1. Show that $e^{-x^2/2}$ is self reciprocal under Fourier cosine Transform.

Soln. :

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c[e^{-x^2/2}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2/2}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2/2}\,\text{R.P. of } e^{isx}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,\text{R.P. of}\int_0^\infty e^{-\frac{x^2}{2}+isx}\,dx$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{1}{2}[x^2-isx+(is)^2-(is)^2]}\,dx$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2-\frac{s^2}{2}}\,dx$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}}\,e^{-s^2/2}\int_0^\infty e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2}\,dx$$

$$t = \frac{x-is}{\sqrt{2}}$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}}\,e^{-s^2/2}\int_0^\infty e^{-t^2}\sqrt{2}\,dt \qquad dx = \sqrt{2}\,dt$$

$$= \text{R.P. of } \frac{2e^{-s^2/2}}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt$$

$$= \text{R.P. of } \frac{2}{\sqrt{\pi}}\,e^{-s^2/2}\,\frac{\sqrt{\pi}}{2} \qquad \because \int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$$

$$F_c[e^{-x^2/2}] = e^{-s^2/2}$$

$\Rightarrow e^{-x^2/2}$ is self reciprocal under Fourier cosine Transform.

Using Parseval's Identity

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\int_{-1}^{1} (1-|x|)^2\, dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2 ds$$

$$2\int_{0}^{1} (1-x)^2\, dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin^2 s/2}{s^2}\right)^2 ds$$

$$= \frac{8}{\pi}\int_{-\infty}^{\infty}\frac{\sin^4 s/2}{s^4}\, ds$$

$$2\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{16}{\pi}\int_{0}^{\infty}\frac{\sin^4 s/2}{s^4}\, ds$$

$$-\frac{2}{3}[0-1] = \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin^4 s/2}{(s/2)^4}\, ds$$

$$\frac{2}{3} = \frac{1}{\pi}\int_{0}^{\infty}\left(\frac{\sin s/2}{s/2}\right)^4 ds$$

Take $t = s/2$

$dt = ds/2 \Rightarrow ds = 2dt$

when

$s=0 \Rightarrow t=0$

$s=\infty \Rightarrow t=\infty$

$$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^4 2\,dt = \frac{2\pi}{3}$$

$$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{2\pi}{6}$$

$$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$

Problems on FT:

4). Find the Fourier transform of $f(x)$ if

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$$

Deduce that i) $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Soln. :

we already find $F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right)$

IFT

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

Problems on Self reciprocal function:

1. Show that the function $e^{-x^2/2}$ is self-reciprocal under Fourier transform.

Soln.:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2 - 2isx + (is)^2 - (is)^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-is)^2 + s^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} e^{-\frac{s^2}{2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2}\, dx$$

Put $t = \frac{x-is}{\sqrt{2}}$

$dt = \frac{dx}{\sqrt{2}}$

$dx = \sqrt{2}\, dt$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \sqrt{\pi} \qquad \because \int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$$

$$F(s) = e^{-s^2/2}$$

Hence

The Fourier transform of $e^{-x^2/2}$ is $e^{-s^2/2}$

Hence $e^{-x^2/2}$ is self reciprocal under F.T.

27. H.W Find the Fourier transform of $f(x) = e^{-x^2}$

2]. Find the Fourier transform of $f(x) = e^{-a^2 x^2}$.

Soln.:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2 x^2 + isx)}\, dx$$

$2axb = isx$, $b = \frac{is}{2a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-s^2/4a^2}\, dx$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2}\, dx$$

Put $t = ax - \frac{is}{2a}$ | $x = -\infty \Rightarrow t = -\infty$

$dt = a\, dx$ | $x = \infty \Rightarrow t = \infty$

$dx = \frac{dt}{a}$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \sqrt{\pi}$$

$$= e^{-s^2/4a^2} \cdot \frac{1}{a\sqrt{2}}$$