Fourier Transform Pair:

The fourier transform of $f(x)$ is gvn. by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \rightarrow (1)$$

Then the function $f(x)$ is the Inverse Fourier transform of $F(s)$ is given by,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \rightarrow (2)$$

The above eqns. (1) and (2) are jointly called Fourier transform Pair.

Self Reciprocal function:

If the fourier transform of $f(x)$ is equal to $F(s)$, then $f(x)$ is said to be reciprocal function under fourier transform.

i.e., $F[f(x)] = F(s)$

Eg: $F[e^{-x^2/2}] = e^{-s^2/2}$

Parseval's Identity or Rayleigh's Theorem:

If $F(s)$ is the fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

Results:

i). $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ ii). $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

Problems :

1. Find the FT of $f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases}$

Soln. :

$-\infty \quad -a \quad a \quad \infty$

$$f(x) = \begin{cases} 1, & -a < x < a \\ 0, & -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_{0}^{a} \cos sx\, dx + i\, 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin sa}{s}$$

2. Find the FT of $f(x) = \begin{cases} x & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$

Soln. :

$$f(x) = \begin{cases} x & \text{if } -a \le x \le a \\ 0 & \text{if } -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} \underbrace{x}_{\text{odd}}\, \underbrace{\cos sx}_{\text{even}}\, dx + i \int_{-a}^{a} \underbrace{x}_{\text{odd}}\, \underbrace{\sin sx}_{\text{odd}}\, dx \right]$$

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$x\cos sx \to$ odd
$x\sin sx \to$ even

$$= \frac{1}{\sqrt{2\pi}}\left[0 + 2i\int_0^a x\sin sx\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{2i\left[x\left(-\frac{\cos sx}{s}\right) - 1\left(-\frac{\sin sx}{s^2}\right) + 0\right]_0^a\right\}$$

$$= \frac{1}{\sqrt{2\pi}}2i\left[\left(-\frac{a\cos sa}{s} + \frac{\sin sa}{s^2}\right) - 0\right]$$

$$= \sqrt{\frac{2}{\pi}}\,i\left[\frac{\sin sa - sa\cos sa}{s^2}\right]$$

3]. Show that fourier transform of

$f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$ and hence find that

$2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$. Hence deduce that

$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)dt = \pi/4$. Using P.I. Show that

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \pi/15$$

Soln. :

$$f(x) = \begin{cases} a^2 - x^2, & -a < x < a \\ 0, & -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2 - x^2)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\underbrace{(a^2 - x^2)}_{\text{even}}\underbrace{\cos sx}_{\text{even}}\,dx + i\int_{-a}^{a}\underbrace{(a^2 - x^2)}_{\text{even}}\underbrace{\sin sx}_{\text{odd}}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_0^a \underbrace{(a^2 - x^2)}_{u} \underbrace{\cos sx}_{v}\, dx + i(0) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(a^2 - x^2) \frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right) - 0 \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[-2x \frac{\cos sx}{s^2} + (a^2 - x^2)\frac{\sin sx}{s} + 2\frac{\sin sx}{s^3} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[-2a \frac{\cos sa}{s^2} + \frac{2 \sin sa}{s^3} \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin sa - sa \cos sa}{s^3} \right]$$

Put $a = 1$, $F(s) = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right]$

i). Using Inverse Fourier Transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right] [\cos sx - i \sin sx]\, ds$$

$$= \frac{2}{\pi}\, 2 \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) \cos sx \, ds$$

Put $x = 0$, $f(0) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\, ds$

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) ds = \frac{\pi}{4} f(0) = \frac{\pi}{4}(1 - 0) = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3} \right) dt = \pi/4$$

ii). Using Parseval's Identity,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

$$\int_{-1}^{1} (1-x^2)^2\,dx = \int_{-\infty}^{\infty} \left\{ 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\right\}^2 ds$$

4×2

$1^2 \quad (x^2)^2$

$a^2 - 2ab + b^2$

$$2\int_{0}^{1} (1 + x^4 - 2x^2)\,dx = \frac{8\times 2}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{5} - 2\frac{x^3}{3}\right]_0^1$$

$$= 2\left[1 + \frac{1}{5} - \frac{2}{3}\right]$$

$$= 2\left[\frac{15+3-10}{15}\right]$$

$$= \frac{16}{15}$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}\left(\frac{\pi}{16}\right)$$

$$\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$

4. Find the FT of $f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

and deduce that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$ and $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt$

Soln. :

$$f(x) = \begin{cases} a-|x|, & -a < x < a \\ 0, & -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} (a-|x|)\,[\cos sx + i\sin sx]\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{a} (a-x)\cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}}\left[(a-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos sx}{s^2}\right]_0^a$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{\cos sa}{s^2} - \frac{1}{s^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos sa}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{2\sin^2 \frac{sa}{2}}{s^2}$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos\theta = 1 - 2\sin^2 \theta/2$

i). Using IFT

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{2}{s^2}\sqrt{\frac{2}{\pi}}\sin^2\frac{sa}{2}\, e^{-isx}\, ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{s^2}\sin^2\frac{sa}{2}\,[\cos sx - i\sin sx]\, ds$$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty}\frac{1}{s^2}\sin^2\frac{sa}{2}\cos sx\, ds$$

Put $x=0,\ a=2$

$$f(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 s}{s^2}\, ds$$

$$\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{4} f(0) = \frac{\pi}{4}(2-|0|)$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2} \qquad \text{'s' is dummy variable.}$$

ii). Parseval's Identity:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-2}^{2} [2-|x|]^2 dx = \int_{-\infty}^{\infty} 4\cdot\frac{2}{\pi}\, \frac{\sin^4 \frac{sa}{2}}{s^4}\, ds$$

$$2\int_0^2 (2-x)^2 dx = \frac{8}{\pi}\, 2\int_0^\infty \left[\frac{\sin \frac{sa}{2}}{s}\right]^4 ds$$

$$2\left[\frac{(2-x)^3}{-3}\right]_0^2 = \frac{16}{\pi}\int_0^\infty \left[\frac{\sin \frac{sa}{2}}{s}\right]^4 ds$$

$$-\frac{2}{3}[0-8] = \frac{16}{\pi}\int_0^\infty \left[\frac{\sin s}{s}\right]^4 ds \qquad \because a=2$$

$$\frac{16}{3}\,\frac{\pi}{16} = \int_0^\infty \left[\frac{\sin t}{t}\right]^4 dt \qquad (\because s \text{ is dummy var.})$$

$$\int_0^\infty \left[\frac{\sin t}{t}\right]^4 dt = \frac{\pi}{3}$$