PARSEVAL'S IDENTITY

For the interval $(-l, l)$, the Parseval's Identity is

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

For the interval $(0, 2l)$

$$\frac{1}{l}\int_{0}^{2l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

For half range cosine series,

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}a_n^2$$

For half range sine series

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2\,dx = \sum_{n=1}^{\infty}b_n^2.$$

1) Find the Fourier series of $f(x) = x^2$ in $-\pi < x < \pi$ and deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

(iv) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$

Soln:- $f(x) = x^2$ in $-\pi < x < \pi$

$f(x)$ is even function

Fourier series is

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx \, dx$$

$u = x^2$ $\qquad$ $v = \cos nx$

$u' = 2x$ $\qquad$ $v_1 = +\frac{\sin nx}{n}$

$u'' = 2$ $\qquad$ $v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0$ $\qquad$ $v_3 = -\frac{\sin nx}{n^2}$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi \frac{\cos n\pi}{n^2}\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

∴ The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx .$$

$$= \frac{\frac{\cancel{2}\pi^2}{3}}{\cancel{2}} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx . \qquad \text{——— (1)}$$

Put $x=0$ in (1)

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos 0$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\cancel{-}\frac{\pi^2}{12} = \cancel{-}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12} \quad (2)$$

Put $x = \pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi$$

$$\pi^2 - \frac{\pi^2}{3} = 4 \left[\frac{(-1)\cos\pi}{1^2} + \frac{(-1)^2 \cos 2\pi}{2^2} + \frac{(-1)^3 \cos 3\pi}{3^2} \cdots \right]$$

$$\frac{3\pi^2 - \pi^2}{3} = 4 \left[\frac{(-1)(-1)}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \cdots \right]$$

$$\frac{\cancel{2}\pi^2}{3} = \overset{2}{\cancel{4}} \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots \right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \quad \text{--- (3)}$$

Adding (2) and (3)

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{12} + \frac{\pi^2}{6}$$

$$= \frac{\pi^2 + 2\pi^2}{12}$$

$$= \frac{\cancel{3}\pi^2}{\cancel{12}\,4}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

$$\frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 16\left[\frac{1}{1^4} + \frac{1}{2^4} + \cdots\right]$$

$$\frac{18\pi^4 - 10\pi^4}{45} = 16\left[\frac{1}{1^4} + \frac{1}{2^4} + \cdots\right]$$

$$\frac{\cancel{8}\pi^4}{45} = \overset{2}{\cancel{16}}\left[\frac{1}{1^4} + \frac{1}{2^4} + \cdots\right]$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \cdots\cdots$$

$$\therefore \frac{1}{1^4} + \frac{1}{2^4} + \cdots\cdots = \frac{\pi^4}{90}.$$