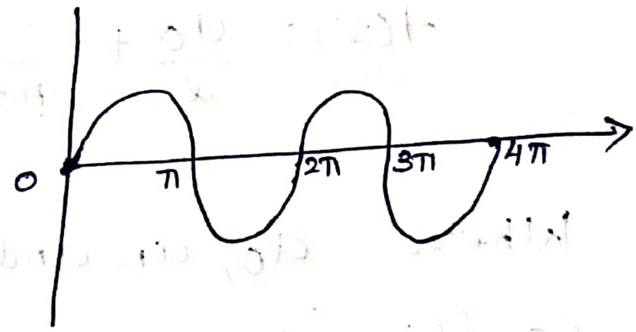
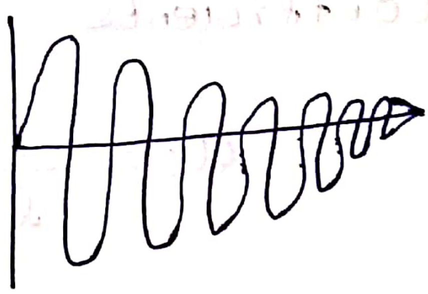
I. Fourier Series

PERIODIC FUNCTION:

A Function $f(x)$ is said to be periodic if $f(x+T) = f(x)$ for all real x and for some positive number T. T is known as period of $f(x)$.

Ex:- $\cos x$, $\sin x$ are all periodic function with period 2π.





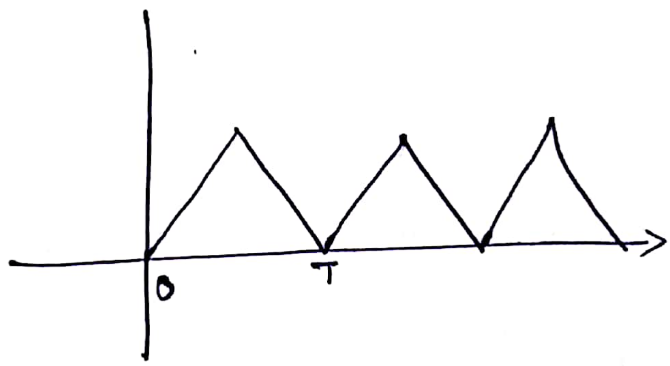
DIRICHLET'S CONDITION :

Any Function f(x) can be developed as a Fourier series provided

(i) f(x) is periodic, ~~Signa~~ single valued and finite.

(ii) f(x) has a finite number of discontinuity in any one period.

(iii) f(x) has a finite number of maxima and minima.



GENERAL FOURIER SERIES

If $f(x)$ is a periodic Function and Satisfies Dirichlet's condition defined for the interval $[c, c+2l)$, then it can be represented by an infinite series is called Fourier Series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Where a_0, a_n and b_n are called Fourier Coefficients.

$$a_0 = \frac{1}{l} \int_{c}^{c+2l} f(x)\,dx.$$

$$a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

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1. Find the Fourier series for the function

$f(x) = x^2$ in $(0, 2\pi)$

Soln:- $f(x) = x^2$ in $(0, 2\pi)$

Fourier series for the function $f(x)$ in $[0, 2\pi)$ is (put $l = \pi$)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx .$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi)^3}{3} - 0\right]$$

$$= \frac{1}{\cancel{\pi}}\left(\frac{8\pi^{\cancel{3}\,2}}{3}\right) = \frac{8\pi^2}{3}$$

$$\boxed{a_0 = \frac{8\pi^2}{3}}$$

To find an:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$u = x^2$ $\quad v = \cos nx$

$u' = 2x$ $\quad v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0$ $\quad v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi}\Big[uv_1 - u'v_2 + u''v_3\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[x^2\frac{\sin nx}{n} - 2x\Big(-\frac{\cos nx}{n^2}\Big) + 2\Big(-\frac{\sin nx}{n^3}\Big)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[0 + \frac{2(2\pi)\cos n2\pi}{n^2} - 0 - 0 - 0 + 0\Big]$$

$$= \frac{1}{\cancel{\pi}}\Big[\frac{4\cancel{\pi}(1)}{n^2}\Big]$$

$$= \quad = \frac{4}{n^2}$$

$$\boxed{a_n = \frac{4}{n^2}}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx.$$

$u = x^2$ $\quad$ $v = \sin nx$

$u' = 2x$ $\quad$ $v_1 = \frac{-\cos nx}{n}$

$u'' = 2$ $\quad$ $v_2 = -\frac{\sin nx}{n^2}$

$u''' = 0$ $\quad$ $v_3 = \frac{\cos nx}{n^3}$

$$= \frac{1}{\pi}\left[uv_1 - u'v_2 + u''v_3\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} - 2x\left(-\frac{\sin nx}{n^2}\right) + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2\cos n2\pi}{n} + 2(2\pi)\frac{\sin n2\pi}{n^2} + \frac{2\cos n2\pi}{n^3} + 0 - 0 - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + 0 + \cancel{\frac{2}{n^3}} - \cancel{\frac{2}{n^3}}\right]$$

$$= \frac{1}{\cancel{\pi}}\left(-\frac{4\pi^{\cancel{2}}}{n}\right]$$

$$\boxed{b_n = \frac{-4\pi}{n}}$$

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$$\therefore \quad f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

$$- \sum_{n=1}^{\infty} \frac{4\pi}{n} \sin nx \quad //.$$