16 MA 201 TRANSFORMS & PARTIAL DIFFERENTIAL EQUATIONS

QUESTION BANK

UNIT – I – FOURIER SERIES

* **Problems under the interval $(0, 2\pi)$:**

1. Find the Fourier Series expansion of $f(x) = x(2\pi - x)$ in $(0, 2\pi)$

2. Determine the Fourier Series for the function $f(x) = x \sin x$ in $0 < x < 2\pi$

3. Find the Fourier Series expansion of $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$

* **Problems under the interval $(-\pi, \pi)$:**

4. Find the Fourier Series expansion of
$$f(x) = \begin{cases} x - 1, & -\pi < x < 0 \\ x + 1, & 0 < x < \pi \end{cases}$$

5. Find the Fourier Series for $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

6. Find the Fourier Series for $f(x) = x^2$ in $(-\pi, \pi)$ and hence deduce that (i) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \infty = \frac{\pi^4}{90}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty = \frac{\pi^2}{6}$ (iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$

7. Expand $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi \end{cases}$ and hence deduce

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}.$$

8. Find the Fourier Series expansion of
$$f(x) = \begin{cases} 1 - x, & -\pi < x < 0 \\ 1 + x, & 0 < x < \pi \end{cases}$$

★ Problems under the interval $(0, 2l)$:

(9) Obtain the Fourier Series of period $2l$ for the function $f(x) = x(2l - x)$ in $(0, 2l)$.

(10) Find a Fourier series with period 3 to represent $f(x) = 2x - x^2$ in $(0, 3)$.

★ Problems under the interval $(-l, l)$:

(11) Obtain the Fourier series expansion of $f(x) = x + x^2$ in $(-2, 2)$ & hence deduce $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty$

★ Half range Fourier Sine and Cosine Series :

(12) Obtain the Fourier cosine Series for

$$f(x) = \begin{cases} kx , & 0 < x < l/2 \\ k(l - x) , & l/2 < x < l \end{cases}$$

(13) Obtain the half range Sine Series for

$$f(x) = \begin{cases} x , & 0 \leq x \leq l/2 \\ l - x , & l/2 \leq x \leq l . \end{cases}$$

(14) Find the half range Sine Series of

$$f(x) = \begin{cases} x , & 0 < x < 1 \\ 2 - x , & 1 < x < 2 \end{cases}$$

(15) Find the half range Fourier cosine Series for $f(x) = x$ in $0 < x < l$. Hence deduce $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \infty$

(16) Find the half range Sine Series for $f(x) = \sin ax$ in $(0, l)$

(17) Obtain the Fourier expansion of $x \sin x$ as a Cosine series in $(0, \pi)$. Hence Show that

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \cdots \infty = \frac{\pi - 2}{4}$$

(18) Find a half range Sine Series of $f(x) = 4x - x^2$ in $(0,4)$. Hence deduce $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots \infty$

(19) Find the half range Sine Series of $f(x) = lx - x^2$ in $(0, l)$.

(20) Obtain the half range Sine and Cosine series of $f(x) = x(\pi - x)$ in $(0, \pi)$.

(21) Obtain the half range Sine Series of $f(x) = (\pi - x)^2$ in $0 < x < \pi$

★ Harmonic Analysis

(22) Compute the first three harmonic of the Fourier Series for $f(x)$ from the following data :

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

(23) Compute upto Second harmonic for the following data :

x :	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
y :	0	9.2	14.4	17.8	17.3	11.7	0

(24) Find the Fourier Series for $y = f(x)$ upto Second harmonic :

x :	0	1	2	3	4	5
y :	9	18	24	28	26	20

UNIT - II FOURIER TRANSFORMS

① Find the Fourier transform of $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$

② Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

Hence deduce that

(i) $\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$

(ii) $\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$

③ Show that the Fourier transform of $f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

is $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$. Hence deduce (i) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

(ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$

④ Obtain the Fourier transform of e^{-ax} and e^{-bx} and hence Show that (i) $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

(ii) $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} =$

using Parseval's identity.

⑤ Evaluate the following integrals using Parseval's identity (i) $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ (ii) $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$

⑥ Show that the transform of $e^{-x^2/2}$ is $e^{-s^2/2}$ by finding the Fourier transform of $e^{-a^2x^2}$, $a \ge 0$.

(7) Find the Fourier transform of $e^{-a|x|}$, $a>0$

and hence deduce that

(i) $\displaystyle\int_0^\infty \frac{\cos xt}{a^2+t^2}\,dt = \frac{\pi}{2a} e^{-a|x|}$

(ii) $F\{x e^{-a|x|}\} = i\sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$

(8) Solve for $f(x)$, the integral equation

$$\int_0^\infty f(x)\sin sx\,dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$$

(9) Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

Hence evaluate (i) $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt$ (ii) $\displaystyle\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

(10) Find the Fourier Sine transform of $f(x) = e^{-ax}$

and hence find Fourier Sine transform $\dfrac{x}{a^2+x^2}$

UNIT III - PARTIAL DIFFERENTIAL EQUATION

1. Form a PDE by eliminating arbitrary function 'f' and 'g' from $z = y f(x) + x g(y)$.
2. Form the PDE by eliminating arbitrary functions f_1, f_2 from $z = x f_1(x+t) + f_2(x+t)$
3. Form the PDE by eliminating arbitrary function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$
4. Form the PDE by eliminating arbitrary function 'f' and 'g' from $z = x^2 f(y) + y^2 g(x)$
5. Form the PDE by eliminating arbitrary function $f(x+y+z, x^2+y^2+z^2) = 0$.
6. Form the PDE by eliminating arbitrary function 'f' and 'g' from $z = f(2x+y) + g(3x-y)$
7. Solve : $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$
8. Solve : $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$
9. Solve : $(x+2z)p + (2xz-y)q = x^2+y$
10. Solve : $(y^2+z^2)p - xyq + xz = 0$
11. Solve : $z(x-y) = x^2 p - y^2 q$.
12. Solve : $(D^2+2DD'+D'^2)z = x^2 y + e^{x-y}$
13. Solve : $(D^2+3DD'+2D'^2)z = \sin(x+5y)$
14. Solve : $(D^2+DD'-6D'^2)z = y\cos x$ (or) $(r+s-6t) = y\cos x$

15) Solve $(D^2 - DD' - 2D'^2)z = 2x + 3y + e^{2x+4y}$

16) Solve : $p^2 + q^2 = z$

17) Solve : $9(p^2 z + q^2) = 4$

18) Solve : $z = px + qy + p^2 + pq + q^2$

19) Solve : $z = px + qy + \sqrt{p^2 + q^2 + 1}$

20) Solve : $x^2 p + y^2 q = z(x+y)$.

UNIT – IV

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATION

1) A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $f(x) = kx(l-x)$ where k is a constant and then released from rest. Find the displacement of any point x of the string at any time $t > 0$.

2) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement y at any time t.

3) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l-x)$, ~~show that~~ find the displacement.

(4) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $V_0 \sin^3\left(\frac{\pi x}{l}\right)$ find the displacement.

(5) A square plate has its faces and its edge $y=0$ insulated. Its edges $x=0$ and $x=10$ are kept at temperature zero and its edge $y=10$ at $100^\circ c$. Find the steady state temperature.

(6) A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. The temperature at short edge $y=0$ is given by,

$$u = \begin{cases} 20x &, 0 \leq x \leq 5 \\ 20(10-x) &, 5 \leq x \leq 10 \end{cases}$$

and all the other three edges are kept at $0^\circ c$. Find the steady state temperature.

(7) An infinitely long rectangular plate with insulated surface is 10 cm wide. The two long edges and one short edge are kept at $0^\circ c$ while the other short edge $x=0$ are kept at temperature

$$u = \begin{cases} 20y &, 0 \leq y \leq 5 \\ 20(10-y), & 5 < y \leq 10 \end{cases}$$

Find the steady state temperature.

(8) A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$, while the other three edges are kept at $0°c$. Find the steady state temperature.

(9) A rod of length l has its ends A and B kept at $0°c$ and $100°c$ until steady state conditions prevail. If the temperature at B is suddenly reduced to $0°c$ and kept so while that at A is maintained, find the unsteady state temperature distribution $u(x,t)$ at a distance x from A and at time t.

(10) A bar 10 cm long, with insulated sides, has its ends A and B kept at $20°c$ and $40°c$ respectively until steady state condition prevail. The temperature at A is then suddenly raised at $50°c$ and at the same instant that B is lowered to $10°c$. Find the subsequent temperature at any point of the bar at any time.

UNIT V

Z-TRANSFORMS

① Find z-transform of (i) $\frac{2n+3}{(n+1)(n+2)}$ (ii) n^2 (iii) $n(n-1)$

② Find z-transform of (i) $\frac{1}{(n+1)(n+2)}$ (ii) na^n (iii) $\cos n\theta$ & $\sin n\theta$

③ Find $z(r^n \cos n\theta)$ and $z(r^n \sin n\theta)$

④ Find inverse z-transform using convolution theorem

(i) $z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$ (ii) $z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$

(iii) $z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$ (iv) $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

⑤ Using z-transform Solve :

(i) $y_{n+2} - 3y_{n+1} - 10y_n = 0$, $y_0 = 1$, $y_1 = 0$

(ii) $y_{n+2} + 2y_{n+1} + y_n = n$, $y_0 = 0 = y_1$

(iii) $y(n+2) - 4y(n) = 2^n$, $y(0) = 0 = y(1)$

(iv) $y(n+3) - 3y(n+1) + 2y(n) = 0$, $y(0) = 4$, $y(1) = 0$, $y(2) = 8$

(v) $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$, $y_0 = 0$, $y_1 = 1$

(vi) $y_{n+2} - 5y_{n+1} + 6y_n = 4^n$, $y_0 = 0$, $y_1 = 1$

⑥ Using complex residue theorem evaluate

(i) $z^{-1}\left[\frac{9z^3}{(3z-1)^2(z-2)}\right]$ (ii) $z^{-1}\left[\frac{10z}{(z-1)(z-2)}\right]$

⑦ Using partial fractions evaluate (i) $z^{-1}\left[\frac{10z}{(z-1)(z-2)}\right]$

(ii) $z^{-1}\left[\frac{z(z^2-z+2)}{(z+1)(z-1)^2}\right]$

⑧ Find z-transforms of (i) $f(n) = \frac{1}{n(n-1)}$ (ii) $f(n) = \frac{a^n}{n!}$