Convolution Theorem of Two functions

The convolution of two functions $f(x)$ and $g(x)$ is defined as,

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Convolution Theorem:

The fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their fourier transforms.

ie, $F[f(x) * g(x)] = F(S) \cdot G(S)$

$= F[f(x)]\, F[g(x)]$

Problems based on Convolution:

I. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transforms.

(or)

Find the Fourier cosine transform of $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$ and Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

(or)

Evaluate parseval's Identity $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

proof: Consider $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

$$F_c[S] = F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$F_c[g(x)] = F_c[S] = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}$$

we know that

$$\int_0^\infty F_c[f(x)]\, F_c[g(x)]\, ds = \int_0^\infty f(x) \cdot g(x)\, dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}\, ds = \int_0^\infty e^{-ax} e^{-bx}\, dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)}\, ds = \int_0^\infty e^{-(a+b)x}\, dx$$

$$\frac{2ab}{\pi} \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty$$

$$= \frac{-1}{a+b} [0-1]$$

$$= \frac{1}{a+b}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

2]. Evaluate $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$ using Transform.

Soln. : consider $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

$$F_s[f(x)] = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$$

$$F_s[g(x)] = F_s[e^{-bx}] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+b^2}$$

Now,

$$\int_0^\infty F_s[f(x)]\, F_s[g(x)]\, ds = \int_0^\infty f(x) \cdot g(x)\, dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+b^2}\, ds = \int_0^\infty e^{-ax} e^{-bx}\, dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \int_0^\infty e^{-(a+b)x}\, dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= -\frac{1}{(a+b)}[0-1]$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \frac{\pi}{2(a+b)}$$

$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{2(a+b)}$$

Parseval's Identity:

1. Using transform methods, evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

Soln.:

consider $f(x) = e^{-ax}$

$$F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

Now, $$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, ds$$

$$\int_0^\infty e^{-2ax}\, dx = \int_0^\infty \frac{2}{\pi} \frac{a^2}{(s^2+a^2)^2}\, ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\frac{-1}{2a}[0-1] = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\frac{\pi}{2a(2a^2)} = \int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

2]. Using transform methods, evaluate

$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$ where $a>0$.

Soln.:

consider $f(x) = e^{-ax}$

$$F_s[f(x)] = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}$$

Now,

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_s(s)|^2\,ds$$

$$\int_0^\infty e^{-2ax}\,dx = \int_0^\infty \frac{2}{\pi}\frac{s^2}{(s^2+a^2)^2}\,ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds$$

$$\frac{-1}{2a}[0-1] = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds$$

$$\Rightarrow \int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds = \frac{\pi}{4a} \Rightarrow \int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a}$$

Find the F.T. of $e^{-a|x|}$ using PI

Show that $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$

Soln.:

$F[e^{-|x|}] = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}$ (Problem 2 Already Proved)

By Parseval's Identity,

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

$$\int_{-\infty}^{\infty} \frac{2}{\pi} \frac{1}{(1+s^2)^2} \, ds = \int_{-\infty}^{\infty} (e^{-|x|})^2 \, dx$$

$$\frac{4}{\pi} \int_0^\infty \frac{ds}{(s^2+1)^2} = 2\int_0^\infty e^{-2x} \, dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_0^\infty$$

$$= -1(0-1)$$

$$= 1$$

$$\Rightarrow \int_0^\infty \frac{ds}{(s^2+1)^2} = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$