Fourier sine Transform:

The fourier sine transform of $f(x)$ is defined by,

$$F_s[s] = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

The inverse fourier sine transform of $F_s(s)$ is given by,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

Fourier cosine Transform:

The Fourier cosine transform of $f(x)$ is defined by

$$F_c[s] = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

The inverse Fourier cosine transform of $F_c(s)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[s] \cos sx \, ds$$

Parseval's Identity:

Sine Transform:

If $F(S)$ is the Fourier transform of $f(x)$, then

$$\int_0^{\infty} [f(x)]^2 dx = \int_0^{\infty} [F_S(S)]^2 ds$$

Cosine Transform:

If $F(S)$ is the Fourier transform of $f(x)$, then

$$\int_0^{\infty} [f(x)]^2 dx = \int_0^{\infty} [F_C(S)]^2 ds.$$

1]. Find the FST of $f(x)$ defined as

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Soln.:

$$F_S(S) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin Sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{1} \sin Sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos Sx}{S} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos S}{S} \right].$$

2]. Find the FST of $1/x$.

Soln.: $\quad F_S(S) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \sin Sx \, dx$

Put $\theta = s x \Rightarrow d\theta = s\,dx$

$$\frac{d\theta}{s} = dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta}\, d\theta$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} \qquad \because \int_0^\infty \frac{\sin\theta}{\theta}\, d\theta = \pi/2$$

$$= \sqrt{\frac{\pi}{2}}$$

3]. Find the FCT of $2e^{-3x} + 3e^{-2x}$

Soln.:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty (2e^{-3x} + 3e^{-2x}) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[2\int_0^\infty e^{-3x} \cos sx\, dx + 3\int_0^\infty e^{-2x} \cos sx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[2\frac{+3}{s^2+3^2} + 3\frac{2}{s^2+2^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{6}{s^2+9} + \frac{6}{s^2+4} \right]$$

4]. Find the ~~FST~~ FCT of e^{-ax} and deduce that using I & P

Soln.: $F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

5]. Find the FCT of $\frac{e^{-ax}}{x}$ and hence, find

$F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

Soln. :

$$F_c[s] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx$$

$$\frac{d}{ds}F_c[s] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\cos sx\right)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}(-x\sin sx)\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\frac{d}{ds}F_c[s] = -\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

~~$F_c(s)$~~

Integrating, we get

$$F_c[s] = -\sqrt{\frac{2}{\pi}}\int \frac{s}{s^2+a^2}\,ds$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int \frac{2s}{s^2+a^2}\,ds$$

$$F_c\left[\frac{e^{-ax}}{x}\right] = -\frac{1}{\sqrt{2\pi}}\log(s^2+a^2)$$

Illy $F_c\left[\frac{e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}} \log(s^2+b^2)$

Now,

$$F_c\left[\frac{e^{ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= \frac{-1}{\sqrt{2\pi}} \log(s^2+a^2) + \frac{1}{\sqrt{2\pi}} \log(s^2+b^2)$$

$$= \frac{1}{\sqrt{2\pi}} \log\left[\frac{s^2+b^2}{s^2+a^2}\right]$$

Hw

1]. Show that $e^{-x^2/2}$ is self reciprocal under FCT.

2]. Find the FST of the function $f(x) = \frac{e^{-ax}}{x}$ and hence find $F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

3]. Find FST of e^{-ax}, $a>0$.

4]. Find the FST of $\frac{x}{x^2+a^2}$

5]. Find the FCT of $f(x) = \begin{cases} \cos x, & 0<x<a \\ 0, & \text{otherwise} \end{cases}$

6]. Show that $e^{-x^2/2}$ is self-reciprocal under FcT.

7. Show that $e^{-x^2/2}$ is self reciprocal under Fourier cosine Transform.

Soln. :

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[e^{-x^2/2}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \text{ R.P. of } e^{isx} \, dx$$

$$= \sqrt{\frac{2}{\pi}} \text{ R.P. of } \int_0^\infty e^{-\frac{x^2}{2} + isx} \, dx$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{1}{2}[x^2 - isx + (is)^2 - (is)^2]} \, dx$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2 - \frac{s^2}{2}} \, dx$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}} \, e^{-s^2/2} \int_0^\infty e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} \, dx$$

$t = \frac{x-is}{\sqrt{2}}$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}} \, e^{-s^2/2} \int_0^\infty e^{-t^2} \sqrt{2} \, dt \qquad dx = \sqrt{2} \, dt$$

$$= \text{R.P. of } \frac{2 e^{-s^2/2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \, dt$$

$$= \text{R.P. of } \frac{2}{\sqrt{\pi}} e^{-s^2/2} \frac{\sqrt{\pi}}{2} \qquad \because \int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$$

$$\therefore [e^{-x^2/2}] = e^{-s^2/2}$$

$\Rightarrow e^{-x^2/2}$ is self reciprocal under Fourier cosine Transform.

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